Composite Fermions And The Fractional Quantum Hall Effect: A Tutorial

I. BACKGROUND

The phenomenon of the fractional quantum Hall effect (FQHE) occurs when electrons are confined to two dimensions, cooled to near absolute zero temperature, and exposed to a strong magnetic field. These lecture notes present an introduction to the essential aspects of the composite fermion (CF) theory of the FQHE, based on a set of lectures at a 2011 summer school in Bangalore, India, which, in turn, borrowed heavily from Ref. 1. The books1–3 and review articles4–9 listed at the end should be consulted for further details, more advanced topics, and references to the original papers.

This section contains some standard results that are needed for an understanding of the FQHE. The missing derivations can be found in the literature.

A. Definitions

We collect here certain relevant units that will be useful below:

\[
\text{flux quantum} = \phi_0 = \frac{hc}{e} \tag{1}
\]

\[
\text{magnetic length} = \ell = \left( \frac{hc}{eB} \right)^{1/2} \approx \frac{25}{\sqrt{B[\text{T}]} } \text{ nm} , \tag{2}
\]

\[
\text{cyclotron energy} = \hbar \omega_c = \hbar \frac{eB}{m_b c} \approx 20B[\text{T}] \text{ K} , \tag{3}
\]

\[
\text{Coulomb scale} = V_C \equiv \frac{e^2}{\epsilon \ell} \approx 50\sqrt{B[\text{T}]} \text{ K} , \tag{4}
\]

\[
\text{Zeeman splitting} = E_Z = 2g\mu_B B \cdot S = \frac{g m_b}{2 m_e} \hbar \omega_c \approx 0.3B[\text{T}] \text{ K}. \tag{5}
\]

The last term in Eq. (2) quotes the magnetic length in nm. The last terms in Eqs. (3), (4), and (5) give the energy (in Kelvin) for parameters appropriate for GaAs (which has produced the best data so far). The magnetic field \(B[\text{T}]\) is in units of Tesla. The Zeeman splitting is defined as the energy required to flip a spin.

B. Landau levels

The Hamiltonian for a non-relativistic electron moving in two-dimensions in a perpendicular magnetic field is given by

\[
H = \frac{1}{2m_b} \left( p + \frac{eA}{c} \right)^2 . \tag{6}
\]

Here, \(e\) is defined to be a positive quantity, the electron’s charge being \(-e\). For a uniform magnetic field,

\[
\nabla \times A = B\hat{z} . \tag{7}
\]

The vector potential \(A\) is a linear function of the spatial coordinates. It follows that \(H\) is a generalized two-dimensional harmonic oscillator Hamiltonian which is quadratic in both the spatial coordinates and in the canonical momentum \(p = -i\hbar \nabla\), and, therefore, can be diagonalized exactly.
We will use the symmetric gauge
\[
A = \frac{B \times r}{2} = \frac{B}{2} (-y, x, 0).
\] (8)

Taking the units of length as the magnetic length \( \ell = \sqrt{\hbar c/eB} = 1 \), and the units of energy the cyclotron energy \( \hbar \omega_c = \hbar c B/m_b c = 1 \), and defining (notice the non-standard definitions)
\[
z = x - iy = re^{-i\theta}, \quad \bar{z} = x + iy = re^{i\theta},
\] (9)
the Hamiltonian becomes:
\[
H = \frac{1}{2} \left[ -4 \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4} \bar{z} \bar{z} - z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right].
\] (10)

We define the following sets of ladder operators:
\[
b = \frac{1}{\sqrt{2}} \left( \frac{\bar{z}}{2} + 2 \frac{\partial}{\partial \bar{z}} \right),
\] (11)
\[
b^\dagger = \frac{1}{\sqrt{2}} \left( \frac{\bar{z}}{2} - 2 \frac{\partial}{\partial \bar{z}} \right),
\] (12)
\[
a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{\bar{z}}{2} - 2 \frac{\partial}{\partial z} \right),
\] (13)
\[
a = \frac{1}{\sqrt{2}} \left( \frac{z}{2} + 2 \frac{\partial}{\partial \bar{z}} \right),
\] (14)
which have the property that
\[
[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1,
\] (15)
and all the other commutators are zero. In terms of these operators the Hamiltonian can be written as
\[
H = a^\dagger a + \frac{1}{2}.
\] (16)

The Landau level (LL) index \( n \) is the eigenvalue of \( a^\dagger a \). The \( z \) component of the angular momentum operator is defined as
\[
L = -\hbar \frac{\partial}{\partial \theta} = -\hbar (b^\dagger b - a^\dagger a) = -\hbar m
\] (17)
with
\[
m = -n, -n + 1, \cdots 0, 1, \cdots
\] (18)
in the \( n \)th Landau level. The application of \( b^\dagger \) increases \( m \) by one unit while preserving \( n \), whereas \( a^\dagger \) simultaneously increases \( n \) and decreases \( m \) by one unit.

The analogy to the Harmonic oscillator problem immediately gives the solution:
\[
H |n, m\rangle = E_n |n, m\rangle
\] (19)
\[
E_n = \left( n + \frac{1}{2} \right)
\] (20)
\[
|n, m\rangle = \frac{(b^\dagger)^m (a^\dagger)^n}{\sqrt{(m+n)!}} \frac{\sqrt{n!}}{|0, 0\rangle},
\] (21)

The single particle orbital at the bottom of the two ladders defined by the two sets of raising and lowering operators is
\[
\langle r | 0, 0 \rangle \equiv \eta_{0,0}(r) = \frac{1}{\sqrt{2 \pi}} e^{-\frac{1}{4}z \bar{z}},
\] (22)
which satisfies
\[ a|0,0\rangle = b|0,0\rangle = 0. \] (23)

The single-particle states are especially simple in the lowest Landau level \((n = 0)\):
\[
\eta_{0,m} = \langle r|0,m\rangle = \left(\frac{b^\dag}{\sqrt{m!}}\right)^m \eta_{0,0} = \frac{z^m e^{-\frac{1}{2} z^2}}{\sqrt{2\pi^2 m!}}.
\] (24)

where we have used Eqs. (12) and (22). Aside from the ubiquitous Gaussian factor, a general state in the lowest Landau level is simply given by a polynomial of \(z\). It does not involve any \(\bar{z}\). In other words, apart from the Gaussian factor, the lowest Landau level (LLL) wave functions are analytic functions of \(z\).

The LL degeneracy can be obtained by considering a region of radius \(R\) centered at the origin, and asking how many states lie inside it. For the lowest Landau level, the eigenstate \(|0,m\rangle\) has its weight located at the circle of radius \(r = \sqrt{2m \cdot \ell}\). Thus the largest value of \(m\) for which the state falls inside our circular region is given by \(m = R^2/2\ell^2\), which is also the total number of eigenstates in the lowest Landau level that fall inside the disk (ignoring order one corrections). Thus, the degeneracy per unit area is
\[
\text{degeneracy per unit area} = \frac{B}{\phi_0} = \frac{1}{2\pi \ell^2},
\] (25)

which is the number of flux quanta penetrating the sample through a unit area.

The filling factor is equal to the number of electrons per flux quantum, given by
\[
\nu = \frac{\rho}{B/\phi_0} = 2\pi \ell^2 \rho,
\] (26)

where \(\rho\) is the 2D density of electrons. The filling factor is the nominal number of filled Landau levels.

C. **Spherical geometry**

F.D.M. Haldane introduced the spherical geometry for the study of the FQHE, wherein the two dimensional sheet containing electrons is wrapped around the surface of a sphere, and a perpendicular (radial) magnetic field is generated by placing a Dirac magnetic monopole at the center of the sphere. The magnetic field
\[
B = \frac{2Q\phi_0}{4\pi R^2} \hat{r},
\] (27)
is produced by the vector potential
\[
A = -\frac{\hbar c Q}{e R} \cot \theta \hat{\phi}.
\] (28)

This geometry has played an important role in testing various theoretical conjectures. Two reasons for the popularity of this compact geometry are: First, it does not have edges, which makes it suitable for an investigation of the bulk properties. Second, Landau levels have a finite degeneracy (for a finite magnetic field), which is useful in identifying incompressible states for finite systems. In particular, filled Landau levels are unambiguously defined. The spherical geometry has been instrumental in establishing the validity of the theory of the FQHE, and provides the cleanest proofs for many properties. We summarize here some essential spherical facts, which will be very useful in the discussion below.

- The magnetic flux through the surface of the sphere, measured in units of the flux quantum, is quantized to be an integer for consistency, denoted by \(2Q\). The quantity \(Q\), called the monopole strength, can be a positive or a negative integer or a half integer. The filling factor is defined by
\[
\nu = \lim_{N \to \infty} \frac{N}{2Q}.
\] (29)
• The orbital angular momentum and its $z$ component are good quantum numbers, denoted by $l$ and $m$, respectively. Their allowed values are:

$$l = |Q|, |Q| + 1, ...$$  \hspace{1cm} (30)

$$m = -l, -l + 1, ..., l.$$ \hspace{1cm} (31)

Each combination occurs precisely once. Note that the minimum value of $l$ is $|Q|$, and $l$ can be either an integer or a half-integer.

• Different angular momentum shells are the Landau levels in the spherical geometry. The degeneracy of each Landau level is equal to the total number of $m$ values, i.e., $2l + 1$, increasing by two units for each successive Landau level. In contrast to the planar geometry, the degeneracy is finite in the spherical geometry, due to its compact nature. For the lowest Landau level, the degeneracy is $2|Q| + 1$; for the next it is $2|Q| + 3$; and so on. The lowest filled Landau level is obtained when $N = 2|Q| + 1$. In general, the state with $n$ filled Landau levels is obtained when $Q$ and $N$ are related as

$$Q = \pm \frac{N - n^2}{2n}.$$ \hspace{1cm} (32)

• The single particle eigenstates are called monopole harmonics, denoted by $Y_{Qlm}(\Omega)$, which are a generalization of the familiar spherical harmonics. (The former reduces to the latter for $Q = 0$.) $\Omega$ represents the angular coordinates $\theta$ and $\phi$ on the sphere. In the lowest Landau level, the monopole harmonics are given by

$$Y_{QQm} = \left[ \frac{2Q + 1}{4\pi} \right]^{1/2} (-1)^{-m} u^{Q-m} v^{-m} Q^m.$$ \hspace{1cm} (33)

with

$$u = \cos(\theta/2)e^{i\phi/2}; \quad v = \sin(\theta/2)e^{-i\phi/2}.$$ \hspace{1cm} (34)

• The product of two single particle wave functions at $Q$ and $Q'$ produces a wave function at $Q + Q'$. In other words, we have

$$Y_{QQm}(\Omega)Y_{QQ'm'}(\Omega) = \sum_{l'} C_{l'l} Y_{Q+Q',l',m+m'}(\Omega).$$ \hspace{1cm} (35)

• Complex conjugation is equivalent to changing the sign of $Q$ (or the magnetic field).

D. General LLL wave function

A many body wave function in the LLL has the form

$$\Psi = F_A[\{z_j\}] \exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right]$$ \hspace{1cm} (36)

where $F_A[\{z_j\}]$ is an antisymmetric polynomial of the $z_j$. The LLL wave function has the property that the polynomial part has no $\bar{z}_j$’s, i.e., it is an analytic function. It is sometimes helpful to view $F_A[\{z_j\}]$ as a polynomial in one of the coordinates, say $z_1$, treating the other coordinates as variables. From the fundamental theorem of algebra, the number of zeroes is determined is equal to the degree of the polynomial, i.e. the largest exponent of $z_1$. That is also the largest occupied orbital. Therefore, the number of zeroes is equal, modulo $O(1)$ terms, to the number of flux quanta penetrating the “sample.” The total number of zeroes of $z_1$ is seen to be equal to $N/\nu$, the angular momentum of the outermost occupied orbital. Furthermore, each zero is actually a vortex, in the sense that a closed loop of $z_1$ around it produces a phase of $2\pi$.

The wave functions that involve the lowest $n$ LLs contain at most $(n - 1)$ factors of $\bar{z}_j$. Because $F_A$ is no longer analytic, no simple statements can be made regarding the number of zeroes, which can now be vortices or anti-vortices.
E. Wave functions for filled Landau levels

For a general filling factor, for non-interacting electrons in the absence of disorder, the ground state is highly degenerate, because electrons in the partially filled topmost Landau level can be arranged in a large number of ways. For an integral filling factor, \( \nu = n \), the ground state is unique, containing \( n \) fully occupied Landau levels. Its wave function is denoted by \( \Phi_n \). In the disk geometry, without a confinement potential there are an infinite number of single particle orbitals; we will view a Landau level as “full” if all states inside a radius \( R \) are occupied.

The wave functions of filled Landau levels (in which all states inside a disk of some radius are filled) are uniquely determined; these are Slater determinants formed from the occupied single particle orbitals. The wave function of the lowest filled Landau level, \( \Phi_1 \) is given by (apart from a normalization factor):

\[
\Phi_1 = \begin{vmatrix}
1 & 1 & 1 & \ldots \\
\bar{z}_1 & \bar{z}_2 & \bar{z}_3 & \ldots \\
\bar{z}_1^2 & \bar{z}_2^2 & \bar{z}_3^2 & \ldots \\
\ldots & \ldots & \ldots & \ldots 
\end{vmatrix} \exp \left[ -\frac{1}{4} \sum_i |\bar{z}_i|^2 \right] 
\]

This is the so-called Vandermonde determinant, which has a particularly simple form:

\[
\Phi_1 = \prod_{j<k} (\bar{z}_j - \bar{z}_k) \exp \left[ -\frac{1}{4} \sum_i |\bar{z}_i|^2 \right] .
\]

F. Integer quantum Hall effect

When plotted as a function of the magnetic field \( B \), the Hall resistance exhibits numerous plateaus. On any given plateau, \( R_H \) is precisely quantized at values given by

\[
R_H = \frac{h}{ne^2},
\]

where \( n \) is an integer. This phenomenon, discovered by Klaus von Klitzing, is referred to as the “integral quantum Hall effect.” The \( R_H = \frac{h}{ne^2} \) plateau occurs in the vicinity of \( \nu \equiv Be/\rho hc = n \), where \( \nu \) is the “filling factor,” i.e., the number of filled Landau levels. In the plateau region, the longitudinal resistance exhibits an Arrhenius behavior:

\[
R_{xx} \sim \exp \left( -\frac{\Delta}{2k_B T} \right) .
\]

This gives an energy scale \( \Delta \), which is interpreted as a gap in the excitation spectrum. \( R_{xx} \) vanishes in the limit \( T \to 0 \), indicating dissipationless transport.

It is customary at first to neglect the Coulomb interaction in the treatment of the integral quantum Hall effect (IQHE). In general, the ground state is highly degenerate, because all arrangements of electrons in the topmost partially occupied LL have the same energy. However, unique ground state solutions are obtained at integer fillings. These are separated from excited states by a gap equal to the cyclotron energy. In other words, the system is incompressible. Including Coulomb interaction does not change this physics so long as the gap does not close, as is the case when the interaction strength is sufficiently small compared to the cyclotron gap.

R. B. Laughlin clarified that the IQHE requires two ingredients: (i) the presence of a gap in a pure system at \( \nu = n \), and (ii) weak disorder. The latter, which is always present in experiment, produces a reservoir of localized states in the gap, needed for the establishment of quantized plateaus.

Exercises

♣ Confirm that the eigenstates for magnetic field pointing in the \(-\hat{z}\) direction are complex conjugates of the above wave functions.

♣ Show that the polynomial part of the single particle wave function (i.e. the factor multiplying the gaussian) of the \( n \)th LL involve at most \( n \) powers of \( \bar{z} \).

♣ Obtain the second LL wave function \( |1, m\rangle \).
Derive Eq. 38.

Show that the wave function of a lowest filled LL with a hole in the $m = 0$ state is given by

$$\Phi_{\text{hole}} = \prod_j z_j \Phi_1$$

(41)

Show that the wave function of a lowest filled LL with an additional electron in the second LL in the $m = -1$ state (smallest angular momentum in the second LL) is given by

$$\Phi_{\text{particle}} = \sum_{i=1}^N \left[ \prod_j (z_i - z_j)^{-1} \right] \bar{z}_i \Phi_1$$

(42)

where the prime denotes the condition $j \neq i$.

Show that the wave function for two filled LLs is given by

$$\Phi_2 = \begin{vmatrix} 1 & 1 & 1 & \ldots \\ z_1 & z_2 & z_3 & \ldots \\ z_1^2 & z_2^2 & z_3^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ z_1^{N/2-1} & z_2^{N/2-1} & z_3^{N/2-1} & \ldots \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 & \ldots \\ \bar{z}_1 \bar{z}_1 & \bar{z}_2 \bar{z}_2 & \bar{z}_3 \bar{z}_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \bar{z}_1 z_1^{N/2-1} & \bar{z}_2 z_2^{N/2-1} & \bar{z}_3 z_3^{N/2-1} & \ldots \end{vmatrix} \exp \left[ -\frac{1}{4} \sum_i \left| |z_i| \right|^2 \right].$$

(43)

Show that

$$\phi_\eta(r) = \frac{1}{\sqrt{2\pi}} \exp \left[ \frac{1}{2} \bar{\eta} \bar{z} - \frac{1}{4} \left| z \right|^2 - \frac{1}{4} \left| \eta \right|^2 \right]$$

is a coherent state, i.e. is an eigenstate of the angular momentum lowering operator $b$. Show that it represents a gaussian localized wave packet at $\eta$. Also show, using ladder operators, that the coherent state in the $n^{th}$ LL is given by

$$\phi^{(n)}(r) \sim (\bar{z} - \bar{\eta})^n \exp \left[ \frac{1}{2} \bar{\eta} \bar{z} - \frac{1}{4} \left| z \right|^2 - \frac{1}{4} \left| \eta \right|^2 \right].$$

(45)

Obtain the solution for an electron in a parabolic confinement potential in the presence of a magnetic field

$$H = \frac{1}{2m_b} \left( p + \frac{e}{c} A \right)^2 + \frac{1}{2} m_b \omega_0^2 (x^2 + y^2)$$

(46)

where $\omega_0$ is a measure of the strength of the confinement. The solutions are known as Fock-Darwin levels.

Show that in the spherical geometry, the wave function of the lowest filled LL is given by

$$\Phi_1 = \prod_{j<k} (u_j v_k - v_j u_k)$$

(47)

Write the explicit wave function $\Phi_1^{\text{hole}}$ in the spherical geometry by leaving vacant the single particle orbital centered at the north pole.

Confirm Eq. 35 for the product of two LLL wave functions at different flux values.

Show that

$$L_+ = -u \frac{\partial}{\partial v}, \quad L_- = -v \frac{\partial}{\partial u}, \quad L_z = \frac{1}{2} \left( u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \right),$$

(48)

satisfy the standard angular momentum algebra, and produce the expected values when applied on $Y_{Q\bar{Q}m}$. 
II. THE FQHE PROBLEM

A. Phenomenology

The phenomenon of the FQHE, discovered by D.C. Tsui, H.L. Stormer and A.C. Gossard, refers to the observation of plateaus in the Hall resistance where it is quantized at

\[ R_H = \frac{h}{f e^2} , \]  

(49)

where \( f \) is a fraction. Beginning with 1/3, more than 70 fractions have been observed to date. The plateau characterized by the fraction \( f \) is centered at the filling factor \( \nu = f \). As in IQHE, the longitudinal resistance exhibits activated behavior in the plateau region, vanishing exponentially as the temperature goes to zero, indicating the presence of a gap in the spectrum.

B. Questions for theory

We will neglect disorder in what follows and assume that, as in IQHE, the presence of a gap at a fractional filling factor \( \nu = f \) for an ideal, disorder-free system, leads to a plateau quantized at \( R_H = h/f e^2 \) with the help of disorder induced localization. The model of non-interacting electrons results in gaps only at integer fillings. We must therefore now deal with the problem of interacting electrons in a strong magnetic field. The gap at fractional fillings must originate from a collective behavior of the electron system.

The goal of theory is to identify the physical mechanism of the FQHE. Stringent constraints are placed on any candidate theory by the following experimental facts:

- Gaps open at a large number of values of the fractional filling factors.
- All fractions in the LLL have odd denominators.
- No FQHE is observed at the simplest fraction \( f = 1/2 \).
- Fractions do not appear in isolation but as members of certain sequences, such as \( f = n/(2n + 1) = 1/3, 2/5, 3/7, 4/9, 5/11, \cdots \).
- FQHE states without full spin polarization can occur at relatively low magnetic fields.
- FQHE is much less prominent in the second LL (which, accounting for spin, occurs in the range \( 2 < \nu < 4 \)). Here, FQHE is also observed at an even denominator fraction, namely \( f = 5/2 \).

In addition to these qualitative features, a great deal of quantitative information has also been gathered (e.g. gaps, collective mode dispersions, spin phase diagram) from both experiments and exact diagonalization studies, that serves as an unbiased benchmark for any theoretical postulate.

C. A nonperturbative many-body problem

The Schrödinger equation for interacting electrons in a magnetic field is given by

\[ H \Psi = E \Psi , \]  

(50)

where

\[ H = \sum_j \frac{1}{2 m_b} \left( \frac{\hbar}{i} \nabla_j + \frac{e}{c} A(r_j) \right)^2 + \frac{e^2}{c} \sum_{j<k} \frac{1}{|r_j - r_k|} + \sum_j U(r_j) + g \mu_B \cdot S . \]  

(51)

\( U(r) \) is a one-body potential incorporating the effects of the uniform positive background and disorder, and the last term is the Zeeman energy. We will consider the limit of large magnetic fields such that

\[ \frac{e^2/\epsilon \ell}{\hbar \omega_c} = \frac{e^2/\epsilon \ell}{E_{\text{Zeeman}}} = 0 \]  

(52)
In this limit all electrons have the same spin and occupy the LLL; their kinetic and Zeeman energies are constant, which we drop. We also forget about disorder, and suppress the background term. In units of the Coulomb energy $e^2/\epsilon \ell$, the Hamiltonian becomes (with $\ell = 1$)

$$H = \sum_{j<k} \frac{1}{|r_j - r_k|} \quad \text{(lowest Landau level)}, \tag{53}$$

which is to be solved in the LLL subspace. We refer to this as the “ideal” limit, because in this limit many parameters that are not relevant to the FQHE physics drop out. In particular, none of the answers in the ideal limit may depend on the electron band mass, which is not a parameter of the ideal Hamiltonian; the energies of the ideal Hamiltonian will depend only on the Coulomb scale $V_C$.

Please, do not let the simplicity of $H$ in Eq. 53 deceive you. This Hamiltonian actually shows that the FQHE problem is a nontrivial one, because it contains no small parameter – in fact, the ideal FQHE Hamiltonian has no parameters, period. To describe a system of interacting electrons we usually begin with a reference state, namely the exact ground state of a suitably chosen $H_0$. For example, Landau’s Fermi liquid theory treats interaction perturbatively around the non-interacting Fermi sea, while the BCS theory finds an instability of the non-interacting Fermi sea state due to a weak attractive interaction. In the case of the FQHE, there is no $H_0$. Switching off the interaction does not provide a unique ground state but an exponentially large number of degenerate ground states. The ground state degeneracy is

$$\left( \frac{N}{\nu} \right)^N \tag{54}$$

which is equal to the number of distinct arrangements of $N$ electrons in $N/\nu$ single particle orbitals. This is of course a very large number.

Experiments are telling us that when the Coulomb interaction is switched on, the exponentially large degeneracy is completely eliminated at some special filling factors to produce a gapped system with a non-degenerate ground state that is some linear combination of the $\left( \frac{N}{\nu} \right)^N$ basis states. The effect of interaction is fundamentally non-perturbative, because the opening of the gap is not related to the strength of the Coulomb interaction, which only sets the energy scale – an arbitrarily weak Coulomb interaction (as can be arranged by taking a very large dielectric constant $\epsilon$) will open a gap.

What physical mechanism underlying this drastic reorganization of the low energy behavior? How does it lead to dramatic phenomena? What else does it produce?

### III. LAUGHLIN’S THEORY

Soon following the discovery of 1/3 FQHE, the first fraction to be observed, Laughlin proposed a wave function for the ground state at $\nu = 1/m$:

$$\Psi^{\text{Laughlin}}_{1/m} = \prod_{j<k} (z_j - z_k)^m \exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right] \tag{55}$$

It has the standard Jastrow form $\prod_{j<k} f(r_j - r_k)$ with pairwise correlations (apart from the gaussian factor); as a result, its probability distribution can be interpreted as the Boltzmann weight of a two-dimensional one component plasma, which allows a derivation of many of its properties. Antisymmetry requires that $m$ be an odd integer, making this wave function applicable to $\nu = 1/3$, 1/5, etc., but not to other fractions. We see below that the 1/m FQHE state is a part of a larger structure that contains other FQHE states as well as physics beyond the FQHE, just as the $\nu = 1$ IQHE state belongs to a larger structure that contains other IQHE states and also a Fermi sea. The reader interested in Laughlin’s motivation for introducing the wave function in Eq. 55, as well as its mapping into a two-dimensional one component plasma, is referred to the literature; this wave function will appear below as the wave function of one filled Λ level of composite fermions.
IV. COMPOSITE FERMION THEORY

The CF theory begins with the presumption that the FQHE and the IQHE are related and can be unified. Experimentally, there is no qualitative distinction between the observations of different plateaus, be they integral or fractional, and therefore it does not seem unreasonable to suspect that their physics might be similar. To be sure, they all arise from the presence of a gap, but we are looking here for a deeper connection.

What would a unification of the FQHE and the IQHE imply? You would think that the FQHE then would be understood as the IQHE of certain emergent fermions. Because IQHE does not require interactions, it follows that these emergent fermions must be weakly interacting. The search for a unified theory thus leads us to suspect the formation of some weakly interacting fermions, which play the same role in the FQHE as electrons do for the IQHE. If such fermions can be identified, they will form the basis for exploring the detailed properties of the many body state, because weakly interacting particles are what we understand best. The emergent fermions of the FQHE state are obviously not electrons, because electrons exhibit only the integral plateaus. What are they, then?

We now know what these fermions are. They are called composite fermions, defined as:

\[ \text{a composite fermion} = \text{an electron} + \text{an even number of quantized vortices} \]  

The vortex, by definition, is an object that produces a phase of \( 2\pi \) for a closed loop around it. A less accurate but more pictorial view of a composite fermion is

\[ \text{a composite fermion} = \text{an electron} + \text{an even number of flux quanta} \]  

where the vortex is modeled as a flux quantum \( \phi_0 \), which also has the property that a closed loop around it produces an Aharonov Bohm phase of precisely \( 2\pi \). Care must be exercised not to take the second definition literally; no real fluxes are bound to electrons. (The bound state of a fermion and a fraction of \( \phi_0 \) was used as a model for “anyon” introduced by J. M. Leinaas, J. Myrheim and F. Wilczek. The braid statistics depends on the value of the bound flux; binding of an even number of flux quanta to fermions gives fermions, but with extra windings.) The composite fermions are depicted as electrons with vertical arrows attached to them, each arrow representing a quantized vortex (or a flux quantum).

A. Accessing FQHE from IQHE through composite fermionization

Why do we define these objects? The motivation comes from the following mean field theory that envisions a scenario for obtaining FQHE states by “composite fermionizing” the IQHE states. I quote here from Ref. 1:

Step I: Let us consider non-interacting electrons at \( \nu^* = n \). The many-particle energy spectrum is shown in the left panel of Fig. (1). The ground state has \( n \) full Landau levels, shown schematically in the left column of Fig. (2 a) for \( \nu^* = 3 \). The lowest energy excited state is a particle-hole pair, or an exciton, shown in the left column of Fig. (2 d). These diagrams have precise wave functions associated with them. We denote the magnetic field by \( B^* \), which can be either positive or negative. It is related to the filling factor by \( \nu^* = \rho\phi_0/|B^*| = n \).

![Diagram of energy spectrum](https://via.placeholder.com/150)

**FIG. 1:** Left: The general structure of the energy spectrum of the many-body system at an integral filling, \( \nu^* = n \). The x-axis label is a convenient quantum number. The CF mean-field theory predicts that the low-energy spectrum at fractional fillings \( \nu = n/(2pn \pm 1) \) has similar structure, except that the states at \( \nu \) are quasi-degenerate and the cyclotron gap evolves into an interaction gap \( \Delta \).

The long range rigidity in the system at an integral filling factor (manifested by the presence of the gap) is caused solely by the Fermi statistics. Thinking in the standard Feynman path-integral language is useful. The partition function gets contributions from all closed paths in the configuration space for which the initial and the final positions
FIG. 2: Panel (a) on the left shows the ground state at $\nu^* = 3$ with three filled LLs. The panels (b), (c) and (d) on the left show a particle, a hole, and a particle-hole excitations. The panels on the right schematically demonstrate the state obtained after the mean field theory, interpreted as the ground state (a) and the CF-quasiparticle, CF-quasihole and CF-quasiparticle quasihole excitations (b, c, and d, respectively.) Horizontal lines in the left column depict Landau levels of electrons, and those on right depict $\Lambda$ levels of composite fermions; the dots are electrons and the dots adorned with two arrows represent composite fermions. The CF diagrams on the right are a pictorial representation of precise microscopic wave functions discussed in the text.

of electrons are identical, although the paths may involve fermion exchanges. Some examples are: a path in which one electron moves in a loop while others are held fixed; or a cooperative ring exchange path in which the closed loop involves many electrons, with each electron moving to the position earlier occupied by the next one. The phase associated with each closed path has two contributions: the Aharonov-Bohm phase which depends on the flux enclosed by the loop; and the statistical phase, which is $+1$ or $-1$, depending on whether the final state is related to the initial one by an even or an odd number of pairwise electron exchanges. The incompressibility at integral fillings is presumably caused by some special correlations built in the phase factors of various paths. These correlations are not easily identifiable in the path integral language, but we know they are present.

Step II: Now we attach to each electron a massless, infinitely thin, massless magnetic solenoid carrying $2p$ flux quanta pointing in $+z$ direction. This converts electrons into composite fermions. The flux added in this manner is unobservable, because it does not alter the phase factors associated with any closed Feynman paths. The excess or deficit of an integral number of flux quanta through any closed path is physically unobservable, and the fermionic nature of particles guarantees that the phase factors of paths involving particle exchanges also remain intact. In other words, the new problem defined in terms of composite fermions is identical to the original problem of non-interacting electrons at $B^*$. We have thus transformed an incompressible state of electrons into an incompressible state of composite fermions.

Step III. This exact reformulation prepares the problem for a mean-field approximation that was not available in the original language. Let us adiabatically (i.e., slowly compared to $\hbar/\Delta$, where $\Delta$ is the gap) spread the flux attached to each electron until it becomes a part of the uniform magnetic field. (Because the initial state has a uniform electron
density, the additional flux, tied to the density, produces a uniform magnetic field.) At the end, we obtain particles moving in an enhanced magnetic field $B$, given by

$$B = B^* + 2pp\phi_0.$$  

(58)

The relation $|B^*| = \rho\phi_0/n$ implies that $B$ is always positive (pointing in the $+z$ direction). The corresponding filling factor is given by

$$\nu = \frac{n}{2pn \pm 1}.$$  

(59)

which follows from the relations $\nu = \rho\phi_0/B$ and $\nu^* = n = \rho\phi_0/|B^*|$. The $+(-)$ sign in the denominator corresponds to $B^*$ pointing in the $+z (-z)$ direction.

Let us now make the crucial assumption that the gap does not close during the flux smearing process, i.e., there is no phase transition. To be sure, quantitative changes will occur. The gap and the wave functions will undergo a complex evolution. (Recall that the IQHE gap is equal to the cyclotron energy, but the FQHE gap may not depend on the electron band mass; the flux spreading will not simply renormalize the value of the gap, but will have to change its energy scale itself.) Nonetheless, if our assumption is correct, then Fig. (1) also represents, qualitatively, the spectrum at $B$.

The absence of a phase transition is an assumption that remains to be verified, and will surely not be valid for all $n$ and $p$. If it is valid for some parameters, however, then the above construction gives a possible way of seeing how a gap can result at the fractions of Eq. (59). Three remarkable features already provide a strong hint that we are on the right track: these fractions are precisely the observed fractions; they have odd-denominators; and we naturally obtain sequences of fractions.

![FIG. 3: CF mean field construction for deriving FQHE from IQHE. We begin with an IQHE state (a); attach to each electron two magnetic flux quanta to convert it into a composite fermion (b); and spread out the attached flux to obtain electrons in a higher magnetic field, which is a FQHE state (c).](image)

The three steps are depicted schematically in Fig. (3). The net effect, in a manner of speaking, is that each electron has absorbed $2p$ flux quanta from the external magnetic field to transform into a composite fermion. Composite fermions experience the residual magnetic field $B^*$.

This can be generalized to arbitrary filling factors. For a general $\nu^*$ we have a band of degenerate ground states for non-interacting electrons. Proceeding as above, we first attach flux quanta to electrons to convert them into composite fermions, and then delocalize the flux to obtain electrons in magnetic field $B = B^* + 2pp\phi_0$. If there are no level crossings, the mean field theory now predicts a quasi-degenerate band of ground states at $B$ that has a one-to-one correspondence with the ground state band at $\nu^*$.

B. Chern-Simons formulation of composite fermions

Chern-Simons approach has been used in the study of anyons (Laughlin); for a bosonic description of the $1/m$ ground state (S.M. Girvin and A.H. MacDonald; S.C. Zhang, T.H. Hansson, and S.A. Kivelson); and for composite fermions (Jain; A. Lopez and E. Fradkin; B.I. Halperin, P.A. Lee and N. Read).

We started above with electrons at $B^*$, added $2p$ flux quanta to each electron, and then made a mean-field approximation to end up with fermions at $B$. In a time-reversed approach, we begin with electrons at $B$, attach $2p$ flux quanta pointing in the direction opposite to $B$, and then perform a mean field approximation to cancel part of the external field, producing fermions at $B^*$ in the end. We consider the Schrödinger equation

$$\left[\frac{1}{2m_b} \sum_i \left(\frac{p_i + e}{c} A(r_i)\right)^2 + V\right] \Psi = E\Psi,$$

(60)
where $V$ is the interaction. Through an exact singular gauge transformation defined by

$$\Psi = \prod_{j<k} \left( \frac{z_j - z_k}{|z_j - z_k|} \right)^{2p} \Psi_{CS},$$

(61)

known as the Chern-Simons transformation, the eigenvalue problem can be expressed as

$$H'\Psi_{CS} = E\Psi_{CS}$$

(62)

$$H' = \left[ \frac{1}{2m_b} \sum_i \left( p_i + \frac{e}{c} A(r_i) - \frac{e}{c} a(r_i) \right)^2 + V \right]$$

(63)

$$a(r_i) = 2p\phi_0 \frac{1}{2\pi} \sum_j \nabla_i \theta_{ij}$$

(64)

The vector potential $-a$ amounts to attaching a point flux of strength $-2p\phi_0$ to each electron (exercise). The Lagrangian corresponding to this Hamiltonian contains the familiar Chern-Simons term $\epsilon_{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha$ for $2+1$ dimensions; $a$ is the Chern-Simons vector potential.

Further progress is not possible without making approximations. The usual approach is to make a “mean-field” approximation, which amounts to spreading the CS flux on each composite fermion into a uniform CS magnetic field, and assuming that, to zeroth order, the particles respond to the sum of CS and external fields. Formally, we write

$$A - a = A^* + \delta A,$$

(65)

$$\nabla \times A^* = B^* \hat{z},$$

(66)

$$B^* = B - 2p\rho\phi_0,$$

(67)

where $B^*$ is the effective magnetic field experienced by composite fermions. The transformed Hamiltonian can now be written as:

$$H' = \frac{1}{2m_b} \sum_i \left( p_i + \frac{e}{c} A^*(r_i) \right)^2 + V + V' = H'_0 + V + V'. $$

(68)

$V'$ contains terms proportional to $\delta A$. The solution to $H'_0$ is trivial, describing free fermions in an effective magnetic field $B^*$. We have thus decomposed the Hamiltonian into two parts: $H'_0$ can be solved exactly and the remainder, $V + V'$, is to be treated perturbatively.

Let us neglect $V + V'$ for a moment. Then, specializing to $\nu = n/(2pn + 1)$, we get eigenfunctions

$$\Psi^*_\nu(B) = \prod_{j<k} \left( \frac{z_j - z_k}{|z_j - z_k|} \right)^{2p} \Phi^*_\nu(B^*),$$

(69)

where $\Phi^*_\nu$ are the solutions of non-interacting fermions at $B^*$ labeled by index $\alpha$, with energies the same as those at $B^*$; in particular, the gap is given by

$$\Delta = \hbar \omega^*_n = \hbar \frac{eB^*}{m_b c} = \frac{\hbar}{(2pn + 1)} \frac{eB}{m_b c},$$

(70)

with $B^* = B/(2pn + 1)$. This solution is far from the actual one. The actual energy gap is proportional to $e^2/\ell \epsilon \sim \sqrt{B}$, the only energy scale in the LLL problem, and is independent of the electron mass, which is not a parameter of the LLL Hamiltonian. The mean-field wave function does not build favorable correlations between electrons; it actually has the same probability amplitude as the uncorrelated IQHE wave function $\Phi_n$. $\Psi_{\text{MF}}$ also involves significant mixing with higher Landau levels, and at $\nu = 1/m$, $m$ odd, it does not reduce to Laughlin’s wave function. The perturbation theory program faces the formidable challenge, without a small parameter, of producing repulsive correlations and of getting rid of the electron mass. It is not known what Feynman diagrams will accomplish that. How do we proceed further, then?

B. I. Halperin, P. A. Lee and N. Read replace the electron band mass by an effective mass, which is to be determined from other considerations and interpreted as the mass of composite fermions, and treat the above theory as an effective model of composite fermions. We refer the reader to Refs. 6 and 7 for a review and references.
C. Wave functions for composite fermions

Another way to make further progress is to construct trial wave functions for composite fermions based on the above physics. The problems mentioned above can be redressed to a large extent by boldly throwing away the denominator on the right hand side of Eq. 69: the new wave function has good correlations, produces Laughlin’s wave function at $\nu = 1/m$, and resides predominantly in the LLL. Because the denominator does not produce any Berry phases, the removal of the denominator does not alter the topological structure of the wave function; it simply converts the point flux quanta bound to electrons into vortices. In what follows, we will present this as our starting postulate, and then explore its various consequences.

This is the CF postulate:

To minimize the interaction energy, each electron in the LLL captures an even number $(2p)$ of quantum mechanical vortices to transform into entities called composite fermions, which themselves can be treated as weakly interacting fermions to a good first approximation.

In other words, the only essential role of interactions is to create composite fermions. As seen below, due to the Berry phases originating from the bound vortices, composite fermions experience an effective magnetic field

$$ B^\ast = B - 2p\rho\phi_0 $$

They form Landau-like levels in the reduced magnetic field, called $\Lambda$ levels, and their filling factor $\nu^\ast = \rho\phi_0/|B^\ast|$ is given by

$$ \nu = \frac{\nu^\ast}{2p\nu^\ast \pm 1} $$

The system of strongly interacting electrons in magnetic field $B$ thus transforms into a system of weakly interacting composite fermions in an effective magnetic field $B^\ast$. The magnetic field $B^\ast$ and filling $\nu^\ast$ are a direct consequence of the formation of composite fermions.

Let us now define mathematically what various terms mean in the preceding paragraph. The meaning of electrons capturing vortices implies the following form for the wave function:

$$ \tilde{\Psi}^{\text{CF}} = \Phi \prod_{j<k} (z_j - z_k)^{2p} $$

where $\Phi$ is an antisymmetric wave function for electrons. The Jastrow factor $\prod_{j<k}(z_j - z_k)^{2p}$ ensures that each particle sees $2p$ vortices on every other particle. The bound state of the electron and the vortices is interpreted as a particle, called composite fermion. The “tilde” on $\Psi$ will be removed below after we make a final minor adjustment to the wave function.

Why would electrons capture vortices? It is easy to see that $\tilde{\Psi}^{\text{CF}}$ is very efficient in keeping electrons away from one another, i.e. has good correlations in the presence of repulsive interactions. The probability of finding two electrons at a short distance $r$ vanishes as $r^{2(p+1)}$ for $\tilde{\Psi}^{\text{CF}}$, which is much faster than the $r^2$ behavior required by the Pauli principle. Of course, this is not a proof that binding vortices to electrons is the best way of minimizing interactions; we cannot rule out that nature would find a cleverer way. The validity of this ansatz must be ascertained by testing its consequences.

An important conceptual ingredient is to allow $\Phi$ to occupy higher Landau levels (LLs), i.e., to allow the presence of $\tilde{z}$’s in the wave function. Even though the wave function on the right hand side is not strictly in the LLL, explicit calculations have shown that it is largely in the LLL, and is adequate for many qualitative purposes. However, because the limit of LLL, appropriate in the limit of very large magnetic fields, is a nice and convenient limit to consider, we would also like to have strictly LLL wave functions. We can imagine obtaining these by an “adiabatic continuation” of the above wave function to the LLL, but it is not known how such an adiabatic process can be implemented. Given
that the mixing with higher LLs is small anyway, we assume that the LLL state can be obtained by a straight LLL projection of the wave function, i.e. throwing away the part of this wave function that resides in higher Landau levels. This gives us the “final” expression for the wave function:

$$\Psi_{\nu}^{\text{CF}} = \mathcal{P}_{\text{LLL}} \Phi_{\pm \nu^*} \prod_{j<k} (z_j - z_k)^{2p}$$  \hspace{1cm} (74)$$

where $\mathcal{P}_{\text{LLL}}$ represents LLL projection. We show below that $\nu$ is given by Eq. 72. Eq. 74 thus relates the wave functions of interacting electrons at $\nu$, which we don’t know, with those of non-interacting fermions at $\nu^*$, which we do know. It is important to appreciate that $\Psi^{\text{CF}}$ is not a single wave function, but gives wave functions for ground and excited states at all filling factors $\nu$ by analogy to the corresponding known states at $\nu^*$. The mapping in Eq. 74 assigns precise wave functions to the diagrams such as those shown in the right panels of Fig. 2 in terms of the known wave functions in the left panels. Composite fermions are the building blocks of the FQHE in the same manner as electrons are for the IQHE.

A remarkable feature of $\Psi_{\nu}^{\text{CF}}$ is that it gives the ground state at $|\nu^*| = n$ uniquely

$$\Psi_{\nu}^{\text{CF}} = \mathcal{P}_{\text{LLL}} \Phi_{\pm n} \prod_{j<k} (z_j - z_k)^{2p}$$  \hspace{1cm} (75)$$

with no adjustable parameters (e.g. Fig. 2a for $\nu^* = 3$). The same is true of the lowest neutral excitation, whose wave function is also fixed by symmetry for any given wave vector (Fig. 2d). The formation of composite fermions is such a powerful constraint that once we assume the form of the wave function, the exponentially large degeneracy of the original problem is eliminated and no further variational freedom remains. Could one seriously expect these wave functions to capture the complex correlations of the FQHE state?

The ground state wave function for one filled AL, $\Psi_{1/(2p+1)}^{\text{CF}}$, precisely reproduces Laughlin’s wave function of Eq. 55, with $m = 2p + 1$ (exercise). This wave function has a particularly simple form because the IQHE wave function $\Phi_1$ is simple (Eq. 38), and because no LLL projection is needed. For general fractions, the wave functions $\Psi_{\nu}^{\text{CF}}$ do not have a simple Jastrow form. This, however, is a matter of technical detail to which no conceptual significance ought to be attached. To consider an analogous situation, even though the wave function $\Phi_1$ is much simpler than $\Phi_{n \neq 1}$, all IQHE states have the same physics: they are states in which an integer number of LLs are fully occupied. Similarly, all $\Psi_{\nu}^{\text{CF}}$ describe states of fully occupied ALs, and are thus on an equal conceptual footing.

The relation between $\nu$ and $\nu^*$ (Eq. 72), or the one between $B$ and $B^*$ (Eq. 71), can be obtained in several ways. The simplest is power counting. Defining the size of the disk by the largest occupied orbital, which has an angular momentum $m_{\text{max}}$, we have $m_{\text{max}} + 1$ flux quanta passing through the disk. The inverse filling factor is then given by $\nu^{-1} = m_{\text{max}}/N$, neglecting O(1) terms. For $\Psi_{\nu}$, the largest occupied angular momentum is given by $2p(N-1)\pm \nu^*$, with the two terms on the right coming from the Jastrow factor and $\Phi_{\pm \nu^*}$. This gives the filling factor in Eq. 72. The magnetic field dependence comes through the magnetic length in the gaussian part of $\Phi_{\pm \nu^*}$. If we keep the magnetic length fixed, the size of the QHE droplet increases upon multiplication by the Jastrow factor to produce a state with smaller density and thus a smaller filling factor.

Next we show that the effective magnetic field for composite fermions results because the Berry phases produced by the bound vortices partly cancel the Aharonov-Bohm phases due to the external magnetic field. For this purpose, we need to calculate the Berry phase of a vortex defined by the wave function

$$\Psi_{\eta} = N_R \prod_j (z_j - \eta)\Psi ,$$  \hspace{1cm} (76)$$

where $\Psi$ is the wave function for the incompressible ground state in question and $\eta = Re^{-i\theta}$ is the location of the vortex. Electrons avoid the point $\eta$, creating a hole there, which has a positive charge relative to the incompressible state. The normalization factor $N_R$ depends on the amplitude of $\eta$, but can be chosen to be independent of the angle $\theta$. Let us now take $\eta$ in a circular loop of radius $R$ by slowly varying $\theta$ from 0 to $2\pi$ while holding $R$ constant. (We are treating $\eta$ as a parameter in the Hamiltonian, which could be an impurity potential located at $\eta$ that binds the
Following Arovas, Schrieffer and Wilczek, the Berry phase associated with this path is given by

\[
\gamma = \oint dt \left\langle \Psi_\eta | i \frac{d}{dt} \Psi_\eta \right\rangle
\]

\[
= \oint d\theta \left\langle \Psi_\eta | i \frac{d}{d\theta} \Psi_\eta \right\rangle
\]

\[
= \oint d\theta (-i) \frac{d\eta}{d\theta} \left\langle \Psi_\eta | \sum_j \frac{1}{z_j - \eta} | \Psi_\eta \right\rangle
\]

\[
= \oint (-i) d\eta \int d^2r \frac{1}{z - \eta} \left\langle \Psi_\eta | \hat{\rho}(r) | \Psi_\eta \right\rangle
\]

\[
= 2\pi \int_{r<R} d^2r \rho_\eta(r)
\]

\[
= 2\pi N_{\text{enc}}.
\]  

(77)

In the above, we have used: \( \hat{\rho}(r) = \sum_j \delta^{(2)}(r_j - r) \), \( \rho_\eta(r) = \left\langle \Psi_\eta | \hat{\rho}(r) | \Psi_\eta \right\rangle \), and \( N_{\text{enc}} \) is the number of particles inside the closed loop. (Because of the definition \( z = x - iy \), the residue for a contour integral differs from the usual by a sign.) The Berry phase of a vortex thus simply counts the number of particles inside the loop, with each particle contributing \( 2\pi \).

The net phase associated with a closed loop traversed by a composite fermion enclosing an area \( A \) is given by

\[
\Phi^* = -2\pi \left( \frac{BA}{\phi_0} - 2pN_{\text{enc}} \right)
\]  

(78)

where the first term is the Aharonov-Bohm phase of the electron and the second term is the Berry phase of the \( 2p \) vortices bound to the composite fermions. For uniform density states, we replace \( N_{\text{enc}} \) in Eq. 78 by its average value \( \rho A \), where \( \rho \) is the electron or the CF density (this is a mean field approximation), and equate the entire phase to the \( \text{AB phase} \) - \( 2\pi B^*A/\phi_0 \) due to an effective magnetic field \( B^* \). That produces the relation Eq. 71. The effective magnetic field thus originates because the Berry phase due to the bound vortices partly cancel the Aharonov-Bohm phase due to the external magnetic field. We have derived this relation for the unprojected wave functions, but it should carry over to the projected wave function if our assumption of adiabatic continuity is valid; direct numerical calculation of the Berry phase have confirmed Eq. 78 for the projected wave functions.

It should be noted that the term vortex here is being used in a different sense than in a superconductor. For the latter, it is the order parameter field that supports a vortex, which has associated with it circulating currents that produce a localized magnetic flux of \( \hbar c/2e \). For composite fermions there is no order parameter; the vortex structure belongs to the microscopic wave function, and has no magnetic flux or circulating currents associated with it. Also, the partial cancellation of the external magnetic field is not caused by an induced current (as for Meissner effect in a superconductor), but by induced Berry phases, and has a purely quantum mechanical origin. The cancellation is internal to composite fermions. An external magnetometer will assess the full applied magnetic field. The only way to measure the effective magnetic field is to use composite fermions themselves for the measurement.

**D. Composite fermions on a sphere**

We construct the wave function \( \Phi_{Q^*} \) of non-interacting electrons at an effective flux \( Q^* \), and multiply by a Jastrow factor to obtain the wave function of composite fermions. The Jastrow factor of the planer geometry translates into

\[
\prod_{j<k} (z_j - z_k)^{2p} \prod_{j<k} (u_jv_k - v_ju_k)^{2p} = \Phi_1^{2p}
\]  

(79)

where \( \Phi_1 \) is the wave function of the lowest filled LL. With LLL projection, this produces

\[
\Psi_{Q^*}^{\text{CF}} = \mathcal{P}_{\text{LLL}} \Phi_{Q^*} \prod_{j<k} (u_jv_k - v_ju_k)^{2p} = \mathcal{P}_{\text{LLL}} \Phi_{Q^*} \Phi_1^{2p}
\]  

(80)

As noted earlier, the monopole strength of the product is the sum of monopole strengths. The monopole strength for \( \Phi_1 \) is \( Q_1 = (N - 1)/2 \), because the LLL degeneracy here is \( 2Q_1 + 1 = N \), which gives

\[
Q^* = Q - p(N - 1)
\]  

(81)
This equation is the spherical analog of Eq. 71: $B^* = B - 2p\phi_0$. With $\nu = \lim_{N \to \infty} N/2Q$ and $\nu^* = \lim_{N \to \infty} N/2|Q^*|$, this relation reduces to Eq. 72 in the thermodynamic limit.

As in the planer geometry, we can construct wave functions for ground states as well as excitations. In the spherical geometry the exact many particle eigenstates are angular momentum multiplets. A nice property of Eq. 80 is that it preserves the angular momentum quantum numbers in going from $Q^*$ to $Q$.

Proof: To see this, let us assume that $\Phi_{Q^*}$ is an eigenstate of the total angular momentum operators $L^2$ and $L_z$ with eigenvalues $L(L + 1)$ and $M$. Write

$$L^2 = L_z^2 + \frac{1}{2}(L_x^2 + L_y^2)$$

(82)

where $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$. The wave function $\Phi_1$ has $L = 0$, so it satisfies

$$L_2\Phi_1 = L_+\Phi_1 = L_-\Phi_1 = 0.$$  

(83)

Noting that all $L_0$, $L_+$, and $L_-$ involve at most first order derivatives, we can commute them through the factor $\Phi_1$. Thus,

$$L_2\Phi_1^2\Phi_{Q^*} = \Phi_1^2L_2\Phi_{Q^*} = M\Phi_1^2\Phi_{Q^*}.$$  

(84)

$$L^2\Phi_1^2\Phi_{Q^*} = \Phi_1^2L^2\Phi_{Q^*} = L(L + 1)\Phi_1^2\Phi_{Q^*}.$$  

(85)

That the angular momentum is not altered upon projection into the lowest Landau level is seen most straightforwardly by writing the projection operator, following Rezayi and MacDonald, as

$$P_{LLL} = \prod_{l=1}^{N} P_{iL}^{l}_{LLL}, \quad P_{LLL}^l = \prod_{l=Q+1}^{\infty} \frac{l(l+1) - L_i^2}{l(l+1) - Q(Q+1)}$$

(86)

where $L_i^2$ is the angular momentum operator for the $i^{th}$ electron. That $P_{LLL}$ indeed is the projection operator is seen by noting that $P_{iL}^{l}_{LLL}$ produces a zero when applied to any single particle state in a higher Landau level, and one when applied to any state in the lowest Landau level. Given that the total angular momentum operators $L^2$ and $L_z$ commute with the $L_i^2$ of an individual electron, they also commute with the projection operator $P_{LLL}$.

The state with an integer number of filled ALs has $L = 0$, i.e. is invariant under rotation. Rotational invariance in the spherical geometry is equivalent to translational invariance in the planar geometry. This implies, in particular, that the state $\Psi_{CF}^L$ with $L = 0$ has uniform density. (This is one example of how the spherical geometry makes many properties essentially obvious: the demonstration of uniform density for $\Psi_{CF}^L$ in the planar geometry requires some work.)

E. LLL projection

The LLL projection is given by (Exercise)

$$\mathcal{P}_{LLL} e^{-\frac{i\pi}{4}\bar{z}}z^m \bar{z}^s = e^{-\frac{i\pi}{4}\bar{z}} \left(2 \frac{\partial}{\partial \bar{z}}\right)^m \bar{z}^s$$

(87)

In other words, to project a wave function into the LLL requires us to write it in a “normal ordered” form by bringing all $\bar{z}$’s to the left of the $z$’s, and making the replacement

$$\bar{z} \rightarrow 2\frac{\partial}{\partial \bar{z}}$$

(88)

with the understanding that the derivatives do not act on the gaussian part. Eq. 87 can be proved by noting that by rotational symmetry the only LLL state into which the projection is nonzero is $\eta_{0,m-s}$ and evaluating its coefficient by using the orthonormality of the single particle orbitals. The LLL projected CF wave functions can thus be explicitly written as

$$\Psi_{CF}^L = \mathcal{P}_{LLL} \Phi_{\pm p^*} \prod_{j<k} (z_j - z_k)^{2p} =: \Phi_{\pm p^*} (\bar{z} \rightarrow 2\partial/\partial \bar{z}) : \prod_{j<k} (z_j - z_k)^{2p}$$

(89)
Here : is the normal ordering symbol, and the derivatives do not act on the gaussian part. (This method for projection is not practical for very large systems; there a slightly different prescription for obtaining LLL states proves more useful, but that is a technicality we will not go into here.) The same prescription works for the projection of an operator \( V(z, \bar{z}) \):

\[
P_{\text{LLL}} V(\bar{z}, z) P_{\text{LLL}} = V_p(\bar{z}, z) = \frac{1}{\sqrt{2}} \, \partial \bar{z}
\]

(90)

Analogous expressions can be obtained for LLL projection in the spherical geometry, which we will not show explicitly here.

F. Energetics: CF diagonalization

For the most interesting states, namely the ground states and low energy excitations at \( \nu = n/(2p \pm 1) \), the wave functions \( \Psi_{\text{CF}} \) are determined completely by symmetry (examples are given below). The predictions for the ground state energy and the CF exciton dispersion are obtained by evaluating the expectation value of the Hamiltonian with respect to these wave functions. Because these are strictly in the LLL, the kinetic energy part is the same for all states at a given \( \nu \) and the energy differences depend only on the Coulomb scale \( V_c \), as must be the case in the LLL.

In general, for \( \nu^* \neq n \), the CF construction gives several states in the lowest energy band, which are treated as correlated basis functions. The CF spectrum is obtained by diagonalizing the Coulomb interaction in this basis. The simplification is that the dimension of the CF basis is exponentially small compared to that of the full LLL basis. The complication is that the CF basis functions are rather complicated, and also not necessarily orthogonal. The evaluation of the CF spectrum requires what is known as “CF diagonalization,” which involves LLL projection, Gram-Schmidt orthogonalization, evaluation of the Hamiltonian matrix, and diagonalization. This can be done exactly for small systems. Efficient Monte Carlo methods have been developed for CF diagonalization for fairly large \( N \), which give the CF spectra and CF eigenstates with high precision. Through these methods the CF theory allows quantitative calculations on rather large systems, involving up to 50-200 composite fermions depending on the filling factor, which enables reliable estimates of the thermodynamic values for various gaps, dispersions, and phase transitions.

Exercises

♠ Show that the vector potential \( a = (\phi/2\pi) \nabla \theta \) produces a flux tube of strength \( \phi \) at the origin, i.e. \( \nabla \times a = \phi \delta(r) \).

♦ Evaluate the Berry phase for the localized wave packet given in Eq. 44 for a circular loop and show that it is equal to the Aharonov bohm phase.

♣ Show that the ground state wave function of one filled \( \Lambda \) level of composite fermions \( (\nu^* = 1) \) is identical to Laughlin’s wave function of Eq. 55 at \( \nu = 1/(2p + 1) \). No LLL projection is needed in this case. We derived the wave functions for the hole and particle of \( \nu = 1 \) state. Construct the corresponding wave functions for the CF-quasihole and the CF-quasiparticle at \( \nu^* = 1 \). (The wave function for the CF quasihole at \( 1/m \) reproduces Laughlin’s wave function for the quasihole.) Then go ahead and construct the wave functions for two CF-quasiholes and two CF-quasiparticles in the innermost angular momentum states.

♣ Prove that \( V \) and \( V_p \) in Eq. 90 have identical matrix elements within the LLL, but when applied to a LLL state, \( V_p \) causes no mixing with higher LLs.

♠ Show that \( x_p \) and \( y_p \), the projected coordinates, obey the commutator

\[
[x_p, y_p] = i\ell^2
\]

(91)

The LLL space is said to be non-commutative.

♣ Prove Eq. 87 by two methods. (i) Noting that the projection is nonzero only for the LLL orbital with angular momentum \( s - m \), evaluate its coefficient by using completeness relation. (ii) Consider \( \tilde{z}^m \phi \), where \( \phi \) is an arbitrary lowest Landau level wave function. Express \( \tilde{z} \) in terms of ladder operators and show that the LLL projection produces \( (\sqrt{2} \hbar)^m \phi \).
V. CONSEQUENCES OF COMPOSITE FERMIONS

A natural first assumption is to neglect the interaction between composite fermions. All states of non-interacting composite fermions can be immediately enumerated.

A. Phenomenology

Due to the formation of composite fermions, the LLL splits into Λ levels of composite fermions that are analogous to LLs of electrons in an effective magnetic field. Gapped states are obtained when the CF filling is integer, i.e. \( \nu^* = n \) (Fig. 4), which corresponds to the electron filling factors given by

\[
\nu = \frac{n}{2pm \pm 1} \quad (92)
\]

Within the LLL, we can equally well express the original problem in terms of holes by using the exact particle hole symmetry. Composite fermionization of holes produces hole filling factors of Eq. 92, which correspond to electron filling factors given by

\[
\nu = 1 - \frac{n}{2pm \pm 1} \quad (93)
\]

In other words:

\[
\text{FQHE of electrons} = \text{IQHE of composite fermions} \quad (94)
\]

At \( \nu = 1/2 \), where \( B = 2pp\phi_0 \), composite fermions experience no effective magnetic field, i.e. we have \( B^* = 0 \). Composite fermions form a Fermi sea here (Halperin, Lee and Read):

\[
\frac{1}{2} \text{ filled LL} = \text{CF Fermi sea} \quad (95)
\]

Because the Fermi sea has no gap, no FQHE results here.
From a microscopic perspective, the CF theory makes detailed quantitative predictions about the low energy sector of the energy spectrum of interacting electrons in the LLL. To summarize:

The CF theory predicts that the exact spectrum of interacting electrons contains a low energy band of states separated from the rest by a gap; further it predicts the quantum numbers, wave functions, and energies, for all states in this band, at all filling factors, by analogy to the low energy band of noninteracting fermions at the corresponding effective filling factor.

This is best illustrated by taking two examples. The spherical geometry is the most convenient for this purpose.

**Example (i):** Consider a system of \( N = 10 \) electrons at flux \( 2Q = 21 \). Here, in the absence of interactions we have \( \binom{22}{10} \) degenerate ground states, which can be chosen as angular momentum eigenstates, with \( M_L \) degenerate multiplets at each allowed \( L \). The structure of the Hilbert space is given by

\[
(N, 2Q) = (10, 21) : \{ L^M_L \} = 0^{52}, 1^{83}, 2^{179}, 3^{212}, 4^{304}, 5^{328}, 6^{418}, \ldots, 60^{1} \text{ (noninteracting electrons)}. \tag{96}
\]

As explained below, the CF theory predicts that with interactions the low energy part of the spectrum splits into mini bands of composite fermions with the structure:

\[
(N, 2Q) = (10, 21) : \{ L^M_L \} = 0^{1} \text{ (lowest CF band)}
\]

\[
\{ L^M_L \} = 1^1, 2^1, 3^1, 4^1, 5^1, 6^1 \text{ (first excited CF band - single CF excitons)} \tag{97}
\]

(the reason why 1 is crossed out is explained below). According to Eq. 81, choosing \( 2p = 2 \), this system maps into a system of weakly interacting fermions at \( Q^* = 1.5 \). What do we know about this system? The LLL is an \( l = 3/2 \) shell with \( 2l + 1 = 4 \) orbitals, the second LL is an \( l = 5/2 \) shell with 6 orbitals, the third LL is an \( l = 7/2 \) shell with 8 orbitals, and so on. With 10 particles the ground state is non-degenerate, obtained by filling the lowest two LLs. (This state is thus a finite size representation of \( \nu^* = 2 \) of composite fermions, or \( \nu = 2/5 \) of electrons.) Being a filled shell state, it has total orbital angular momentum \( L = 0 \). The lowest energy excited states are CF excitons, obtained by promoting a fermion from the \( l = 5/2 \) shell to the \( l = 7/2 \) shell. The allowed angular momenta for the exciton thus are \( L = 1, 2, 3, 4, 5, 6 \), with a single multiplet at each of these values. Furthermore, we can write the explicit wave functions of these states at \( Q^* \), from which we can construct \( \Psi^\text{CF} \) for interacting electrons at \( Q \). With the wave functions, the energies are obtained by taking expectation value of the Coulomb interaction; this requires some work, but is in principle straightforward.

Being a single state, the wave function \( \Phi_{Q^*} \) is determined uniquely by group theory for the ground state or the exciton, and hence \( \Psi^\text{CF}_{\text{LLL}} \) has no dependence on the interaction either for these states. As mentioned above, this remains the case for arbitrarily large \( N \) for systems that map into filled shells at the effective flux, which are the most interesting states, namely the incompressible states responsible for the FQHE.

Finally, a somewhat subtle point ought to be noted: some state at \( Q^* \) might not produce a state at \( Q \), because the wave function may be annihilated upon LLL projection. For the lowest energy band, no such annihilations are known to occur; each state at \( Q^* \) produces a state at \( Q \). Such one-to-one correspondence between \( Q^* \) and \( Q \) is lost for higher bands, however (as it must, given that there are only a finite number of LLL states at \( Q \) but an infinite number of state at \( Q^* \) where we allow all LLs). The first example is the \( L = 1 \) exciton state, which is annihilated by LLL projection.

**Example (ii):** Consider next a system of \( N = 8 \) electrons at \( 2Q = 18 \). Here, for non-interacting electrons we have

\[
(N, 2Q) = (8, 18) : \{ L^M_L \} = 0^{13}, 1^{17}, 2^{42}, 3^{75}, 4^{111}, 5^{70}, 6^{91}, \ldots, 44^{1} \text{ (noninteracting electrons)}. \tag{98}
\]

With interactions, composite fermion formation predicts a miniband

\[
(N, 2Q) = (8, 18) : \{ L^M_L \} = 0^{1}, 1^{1}, 2^{1}, 3^{1}, 4^{1}, 6^{1} \text{ (lowest CF band)} \tag{99}
\]

To see how this is arrived at, first notice that the system maps into 8 weakly interacting fermions at \( Q^* = 2 \). The LLL accommodates \( 2Q^* + 1 = 5 \) fermions, which can be treated as inert. We are thus left with three fermions in the angular momentum \( l = 3 \) shell, where they can occupy \( l_z = -3, -2, -1, 0, 1, 2, 3 \) orbitals. The allowed \( L \) values can be obtained from elementary group theory. The simplest method is to list all distinct configurations for various values of \( L_z \) consistent with the Pauli principle. This gives us 5, 4, 4, 3, 2, 1, 1 distinct configurations at \( L_z = 0, 1 \), 2, 3, 4, 5, 6, respectively. Noting that each \( L \) multiplet produces one state at \( L_z = -L, -L+1, \ldots L \) we obtain the allowed \( L \) values quoted in Eq. 99. Again, the wave functions of these states can be constructed (no more than a single state occurs at each \( L \) value at the effective flux \( 2Q^* \), and therefore the wave function \( \Phi_{Q^*} \) is determined at each \( L \) uniquely from group theory alone) and their Coulomb energies evaluated by straightforward (albeit sometimes cumbersome) methods.
Exercises

diamondsuit Obtain the $L$ quantum numbers of states in the lowest energy bands of $(N, 2Q) = (8, 17), (8, 19), (8, 20), (10, 27), (12, 23), (12, 29)$.
diamondsuit Obtain the relation between $N$ and $2Q$ for $\nu = 2/5$ and $3/7$.
diamondsuit The $2/3$ state can be obtained in two ways: from $1/3$ by using particle hole symmetry, or from $\nu = 2$ by composite fermionization with reverse flux attachment. Show that both occur at the same flux and produce the same $L$ values for the exciton. (Explicit calculations show that the wave functions obtained from these two methods are essentially identical, so they represent two equivalent ways of looking at the same state.)
VI. WHY WE BELIEVE IN COMPOSITE FERMIONS

Now that we have carefully defined the CF theory, we must address the all important question: Does it capture the correct physics of the FQHE? Fortunately, the formation of composite fermions produces a fairly extensive interconnected web of qualitative and quantitative consequences. This section deals with three facets of the CF physics, each with numerous ramifications, that were put to the test in the early 1990s for the physics of FQHE in the lowest LL. (This section is concerned with the LLL; higher LLs are discussed in the next section.)

A. Explanation of the FQHE

The understanding of the FQHE of electrons as the IQHE of composite fermions explains the following observations:

• Only odd denominator fractions are obtained.
• Fractions appear in sequences, because they are all derived from the sequence of integers.
• The fractions given by Eq. 92 are indeed the prominent fractions observed in experiments. For example, the data in Fig. 5 show the sequences:

\[
\frac{n}{2n+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots
\]

\[
\frac{n}{2n-1} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \ldots
\]

The fractions belonging to the sequences \( n/(4n+1) \) and \( n/(4n-1) \) are seen at higher magnetic fields, in the range \( 1/3 < \nu \leq 1/5 \).

• The CF theory explains the similarity of the FQHE data, plotted as a function of the effective field, with the IQHE (see Fig. 6). Although obvious in retrospect, it had not been noticed prior to the CF theory. The comparisons show that just as transitions from one integral plateau to another occur as the Fermi level passes through LLs, so do transitions from one fraction to the next as the Fermi level crosses \( \Lambda \) levels.

• The exactness of the Hall quantization results because the number of vortices bound to composite fermions \( (2p) \) and the number of filled \( \Lambda \)Ls \( (n) \) are both integers; consequently, the right hand side of \( f = n/(2pn \pm 1) \) is not susceptible to continuous variations of parameters.
The FQHE and the IQHE are thus unified. We also now have a physical understanding of the origin of gaps in a partially filled Landau level: when composite fermions are formed, the LLL splits into Λ levels (ΛLs) of composite fermions, and incompressible states are obtained when an integral number of ΛLs are full.

![Plot of FQHE data](image)

**FIG. 6:** Plotting FQHE data as a function of the effective magnetic field by shifting the magnetic field axis by 2ρφ0 demonstrates its similarity to the IQHE. Source: L.N. Pfeiffer, W. Pan, R.R. Du, H.L. Stormer and D. C. Tsui.

### B. Explanation of the 1/2 state

The FQHE sequence \( n/(2pn±1) \) terminates into 1/2 in the limit \( n \to \infty \). The state with an infinite number of filled Λ levels is nothing but a Fermi sea of composite fermions, just as an infinite number of filled LLs is another name for the Fermi sea of electrons. The same conclusion is reached by noting that the effective magnetic field \( B^* = B - 2\rho\phi_0 \) vanishes at \( \nu = 1/2 \) because \( B = \rho\phi_0/\nu \). The following observations are explained by the CF Fermi sea\(^{1,4,8,9}\):

- The state remains compressible down to the lowest attainable temperatures.
- The current carrying objects execute semiclassical cyclotron orbits in the vicinity of \( \nu = 1/2 \) that have been measured in several experiments. The measured radius is consistent with
  \[
  R_c^* = \frac{\hbar k_F^*}{eB^*}, \tag{102}
  \]
  with \( k_F^* = \sqrt{4\pi\rho} \), as appropriate for a fully polarized CF Fermi sea. The radius is much larger than that of the orbit an electron would execute in the external magnetic field.
- Quantum oscillations, the telltale signature of a Fermi sea, have been observed, and are well described by standard expressions developed for an ordinary Fermi sea.
- The temperature dependence of the spin polarization of the 1/2 state is consistent with that of a Fermi sea of non-interacting fermions.
C. Explanation of computer experiments

FIG. 7: Top left: Comparison of exact spectra (dashes) and CF spectra (dots) for Coulomb interaction. Top two spectra correspond to \((N, 2Q) = (10, 27)\) and \((8, 21)\); middle two spectra correspond to \((N, 2Q) = (12, 23)\) and \((9, 16)\); and bottom two spectra correspond to \((N, 2Q) = (12, 29)\) and \((9, 16)\). Source: Chapter 1 by J.K. Jain and R.K. Kamilla, in Ref. 2. Top right: Comparisons for larger systems at \(1/3\), \(2/5\) and \(3/7\). The dark dots are exact spectra and the light dots are CF spectra. Source: A. Wójs. Bottom left: Comparison for \((N, 2Q) = (12, 29)\); only the very low energy part of the spectrum is shown. (Source: J.K. Jain, C.-C. Chang, G.-S. Jeon, and M.R. Peterson, Solid State Commun. 127, 809 (2003).) Bottom right: Comparisons for several eight particle systems between \(1/3\) and \(2/5\) (Source: Chapter 1 by J.K. Jain and R.K. Kamilla, in Ref. 2). The number of independent multiplets at various \(L\) values are given at the top axis. With the exception of the labels of the top right panels, all energies shown are energies per particle and include interaction with the background. The CF spectra contain no adjustable parameters, and are determined uniquely from the wave functions of Eq. 75 or 80 without making any approximations.

How good are the microscopic wave functions? We compare in this section results from two independent sets of calculations. (a) One calculation obtains the exact Coulomb spectra and exact Coulomb eigenstates of \(N\) interacting electrons at flux \(2Q\) by a brute force numerical diagonalization. (b) In the second calculation, we obtain the spectra predicted by the CF theory, again without making any approximations. For this purpose, we enumerate all degenerate ground states and their quantum numbers (e.g. the orbital angular momentum) of free fermions at \(2Q^* = 2Q - 2(N-1)\), from which construct the CF wave functions according Eq. 74. When the wave function is uniquely determined, as is the case for the filled \(\Lambda L\) ground state at \(\nu^* = n\) and its CF exciton, all that we need to do is evaluate the expectation value of its Coulomb energy. In general, we perform a CF diagonalization to obtain the CF spectrum. Because both calculations are exact with no adjustable parameters, the comparisons are unbiased. (For \(\Psi^{CF}\) we can obtain the energy either exactly or by the Monte Carlo method for evaluating multi-dimensional integrals; the latter gives results as accurately as we please by running the computer long enough, but we typically settle for four significant figure
We put the exact and the predicted spectra on the same plot in Fig. 7 for 15 different systems labeled by \((N, 2Q)\). The following features can be noted:

- The exact many body spectrum indeed contains a low energy band of states. This low energy structure arises purely due to interactions; all LLL states would be degenerate in the absence of interactions.
- The structure of the low-energy band depends on \(2Q\) and \(N\) in a seemingly haphazard manner, but is seen to have a one-to-one correspondence with that of noninteracting fermions at \(2Q^*\), as predicted by the CF theory. (These quantum numbers were worked out above in Sec. VB and in the exercises.)
- The quantum numbers of the exciton band are also predicted correctly.
- The energies obtained from \(\Psi^{CF}\) are very accurate. They are typically within 0.1% of the exact energy for the systems studied.
- The wave functions \(\Psi^{CF}\) have a near unity overlap with the corresponding exact eigenstates (Table I).

To assess the significance of these comparisons, we note that each eigenstate is a linear combination of a large number of basis functions, shown at the top of several panels, and therefore an agreement for even a single eigenstate is nontrivial, especially in view of the parameter-free nature of the theory. Fig. 7 already shows comparisons for more than 100 eigenstates. These represent typical comparisons. The CF theory has been tested against exact results for a large number of other \((N, 2Q)\) systems in the LLL that are accessible to numerical diagonalization. These studies have demonstrated that the CF theory gives a close to exact description for all incompressible ground states at \(\nu = n/(2pn \pm 1)\), their neutral excitations, and for all lowest band states at all filling factors in between, in the filling factor range where the FQHE is observed; the CF theory does not miss any state, nor does it predict any spurious state. Finally, it is noted that even though \(\Psi^{CF}\) were motivated by a mean field approximation, they turn out to be essentially the exact solution of the problem.

In particular, the excitations of all FQHE states are seen to be simply excited composite fermions: The charged excitations of all \(\nu = n/(2pn \pm 1)\) states are either additional composite fermions in the \((n + 1)\textsuperscript{th}\) A level (see Fig. 2 b) or missing composite fermions in the \(n\textsuperscript{th}\) A level (Fig. 2 c), sometimes referred to as CF quasiparticles or CF quasiholes; and neutral excitations are particle hole pairs of composite fermions, i.e. CF excitons (Fig. 2 d). Figs. 7 and 8 show that the CF description of quasiparticles, quasiholes, and excitons is essentially exact. Even though the CF theory provides a simple mental picture for them, they are objects with complicated density profiles, as seen in
Fig. 8. It should be noted that there is no particle hole symmetry between quasiparticles and quasiholes at a given fraction, because they reside in different \( \Lambda L \)s; also a quasiparticle costs larger energy because of the effective cyclotron energy cost associated with its creation.

TABLE I: Overlaps of the wave functions \( \Psi^{\text{CF}} \) with the corresponding exact (Coulomb) ground state wave functions for several incompressible states. Results are taken from G. Fano, F. Ortolani, and E. Colombo, Phys. Rev. B 34, 2670 (1986) for \( \nu = 1/3 \) (Laughlin); G. Dev and J.K. Jain, Phys. Rev. Lett. 69, 2843 (1992) for \( \nu = 2/5 \); and X.G. Wu, G. Dev, and J.K. Jain, Phys. Rev. Lett. 71, 153 (1993) for \( \nu = 3/7 \) and 2/3.

<table>
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<th>( n )</th>
<th>( N )</th>
<th>overlap</th>
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<td>0.9964</td>
</tr>
<tr>
<td></td>
<td>8</td>
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<tr>
<td></td>
<td>9</td>
<td>0.9941</td>
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</tr>
<tr>
<td>( 2/5 )</td>
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<td>6</td>
<td>0.9998</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.9996</td>
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</tr>
<tr>
<td>( 3/7 )</td>
<td>3</td>
<td>9</td>
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</tr>
<tr>
<td>( 2/3 )</td>
<td>-2</td>
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</tr>
<tr>
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FIG. 9: Comparing CF and exact spectra for states of composite fermions of vorticity \( 2p = 4 \). Source: A. Wójs.

A comparison of the exact and the CF spectra is shown in Fig. 9 for several systems in the range \( 1/3 > \nu \geq 1/5 \), where composite fermions have four vortices bound to them. Even though the CF theory is quantitatively less accurate than it is in the range \( 2/3 > \nu \geq 1/3 \), it is still very accurate (the ground state energies are off by less than \( \sim 0.2\% \)), and captures the qualitative physics. The somewhat larger deviation for \( 1/5 \) is likely due to its proximity to a Wigner crystal (there is evidence for an insulating state, believed to be a Wigner crystal, on either side of 1/5). The anomalously large disagreement between the exact and predicted energies of the CF exciton at 1/5 at small \( L \) arises because the exact state here is made of two CF excitons, with an energy approximately equal to twice the minimum energy of the single exciton.

This string of successes in the early 1990s gave a definitive and indisputable confirmation of composite fermions. It also established a deep and satisfying connection between the FQHE and the IQHE not only at the qualitative level, as seen from the phenomenology and the similarity between the spectra at \( B \) and \( B^* \), but also at the most microscopic level, through a direct relation between their quantum mechanical wave functions. Various previously puzzling facts become evident, and even inevitable, when viewed with the knowledge of composite fermions. The compressible and incompressible states of the LLL are also referred to as CF states.
VII. FURTHER DEVELOPMENTS AND UNRESOLVED ISSUES

The previous section focused on the initial developments in the field. During the last two decades the CF theory has been applied to many other situations. We mention them here briefly, leaving a motivated reader to explore the literature on his or her own.

A. Calculation of thermodynamic quantities

From the wave functions $\Psi^{CF}$ for the ground and excited states, thermodynamic limits can be obtained numerically for various energies as well as equal time correlation functions such as the pair correlation function. The multidimensional integrals are computed numerically using the Monte Carlo method, and good accuracy can be obtained. We expect the results to be accurate to within a few percent of the exact result for the ideal model, as seen in exact diagonalization studies. The qualitative features of these results can be compared to experiment, but for a better quantitative comparison, it is necessary to include the corrections due to LL mixing, nonzero thickness, and disorder, which are all neglected in the “ideal” model. The first two can be incorporated to a reasonable degree, but the quantitative effect of disorder has been difficult to estimate.

B. Spin physics

In our discussion above we have considered fully spin-polarized states, as appropriate in the limit of very large Zeeman energies. Partially spin polarized or spin single FQHE states become possible at low Zeeman energies. Let us assume that composite fermions are formed even at low Zeeman splittings, and can be taken as nearly independent particles. Their IQHE occurs at $\nu^* = n = n_\uparrow + n_\downarrow$, where $n_\uparrow$ and $n_\downarrow$ are the number of occupied spin-up and spin-down $\Lambda$ levels. This corresponds to electron fillings

$$\nu = \frac{n}{2pm \pm 1} = \frac{n_\uparrow + n_\downarrow}{2p(n_\uparrow + n_\downarrow) \pm 1}.$$  \hspace{1cm} (103)

The spin polarization of the state is given by

$$\gamma_e = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow}.$$  \hspace{1cm} (104)

Thus, the FQHE occurs at the same fractions as before (Eq. 92) but many states with different spin polarization are possible at each fraction. The possible states for $\nu = 4/7$ or 4/9 are illustrated in Fig. (10). As the Zeeman energy is lowered, the FQHE system undergoes magnetic phase transitions from one incompressible state to another.

![Fig. 10: Schematic view of the evolution of the FQHE state at $\nu = 4/9$ or 4/7, which maps into $\nu^* = 4$ filled $\Lambda$ levels, as a function of the Zeeman energy, $E_z$. The three different possible states are $(n_\uparrow, n_\downarrow) = (4, 0), (3, 1), \text{ and } (2, 2).$](image-url)

The Zeeman energy can be varied, for example, by application of an additional magnetic field parallel to the layer (i.e. by tilting the sample while holding the perpendicular component of the field fixed), which affects the Zeeman energy but, to the lowest order, not the interaction energy. The above physics has been confirmed by transport and optical experiments. Phase transitions have been observed and the Zeeman splittings where they occur agree well with the theoretical predictions. The spin polarizations have also been measured and agree with the expression in Eq. 104. Interestingly, the spin physics of the FQHE is richer than that of the IQHE (in GaAs) because the Zeeman energy is very small compared to the electron cyclotron energy (thus producing the state with the smallest spin polarization), but it is typically comparable to the FQHE gaps (interpreted as the cyclotron energy of composite fermions).
The results in this section also apply to experimentally observed FQHE in multivalley systems, such as the one in AlAs quantum wells, with the two valleys playing the role of the two spin components. These considerations can be extended to situations where each electron has four components, e.g., graphene where the electron has two valley indices (called K and K’ points) and two spin indices.

C. Charged and neutral excitation gaps

The energy required to create a far separated particle hole pair of composite fermions has been calculated for various FQHE states. This can be compared to the activation energy appearing in the temperature dependence of the resistance. While the observed ordering of the stability of states is consistent with the predicted ordering (larger the gap the more stable the state), the measured gaps are smaller than the predicted ones by up to a factor of two. The measured dispersions of the neutral excitations are in qualitative agreement with the theoretical ones, but again with somewhat lower energies. The quantitative discrepancy is likely caused by disorder, not included in theory. Phenomenologically, subtracting a constant $\Gamma$ from all theoretical gaps produces a reasonable account of the experimental gaps; $\Gamma$ is interpreted as a disorder induced broadening of the ALs.

D. Physics arising from the residual interaction between composite fermions

If composite fermions were strictly non-interacting, all states of the lowest CF band would be degenerate and the CF exciton would be flat. The splitting of states in the lowest band and the dispersion of the CF exciton are a signature of the residual interaction between composite fermions. As noted above, even though $\Psi_{\text{CF}}$ are constructed from the wave functions of non-interacting electrons, they accurately capture the physics of the residual interaction between composite fermions. (Composite fermions are weakly interacting in the sense that the splitting between the states in the lowest band is small compared to the gap separating them from other states.)

New physics can arise from the residual interaction between composite fermions for non-integer values of $\nu^*$, just as new physics arises from Coulomb interaction for non-integer values of $\nu$. Theoretically, a system of interacting composite fermions with $n - 1 < \nu^* < n$ can be treated in two ways. (i) We construct $\Psi_{\text{CF}}$ for a system with $n - 1$ filled ALs and two additional composite fermions in the $n^{\text{th}}$ AL. The energy as a function of the distance between the two composite fermions gives the effective interaction between them. This has a complex form in general, but can be used as an input for the study of a system of many composite fermions in the $n^{\text{th}}$ AL, assuming that the three and higher body interaction is negligible for composite fermions. (ii) A more accurate method is to construct the CF basis for all lowest band state at $\nu^*$ and diagonalize the Coulomb interaction in this basis. This is what was done in the lower left panel of Fig. 7 to obtain the spectrum.

The residual interaction between composite fermions can cause new states with even more complex structures. Here are some examples:

- **FQHE:** In principle, the states $f = n/(4n\pm 1)$, which are interpreted as the IQHE of composite fermions carrying four vortices, can also be viewed as the FQHE of composite fermions carrying two vortices (see Fig. 6). New fractions can also appear. Consider, for example, $\nu^* = 4/3$ (fully spin polarized), where the lowest AL is full and the second is $1/3$ occupied. It is possible that the residual interaction between composite fermions in the second AL is of a form that makes it energetically favorable for composite fermions to capture two more vortices to turn into higher order composite fermions to produce a new FQHE state at $f = 4/11$. Should this happen, this FQHE state will be a “mixed” state containing both composite fermions carrying two vortices (lowest AL) and four vortices (second AL). This bottom left spectrum in Fig. 7 is a finite size representation of $4/11$. The fact that the lowest energy band is well explained by the CF theory indicates that the state can be modeled in terms of composite fermions, and CF diagonalization demonstrates that the residual interaction between them produces a uniform ($L = 0$) ground state separated from others by a gap; as a further confirmation, the ground state has been shown to be very close to the state obtained by composite fermionization of the $4/3$ FQHE. Other fractions can similarly occur. There is experimental evidence for such FQHE states in the neighborhood of $1/3$, which are much more delicate than the principal ones at $n/(2pm \pm 1)$.

- **Pairing:** The FQHE at $\nu = 5/2$ is believed to occur due to a pairing of composite fermions, discussed below, induced by a weak attraction between them.

- **Wigner crystal / stripe:** It is possible that when the density of composite fermions in the topmost AL is small, they may form a Wigner crystal or a bubble phase, depending on the parameters. In half filled higher ALs a stripe phase of composite fermions may be formed.
E. Fractional local charge

The charge access or deficiency associated with a charged excitation, called its local charge, is a precise fraction of the electron charge, as first demonstrated by Laughlin in the context of $f = 1/m$ FQHE. We derive below the fractional charge by several methods. Even though the excitations of all FQHE states are composite fermions, their local charge will not be the same because of the different screening properties of the FQHE states.

**Method 1:** To stress that the fractional charge is a direct consequence of incompressibility at a fractional filling factor, let us first give, following Laughlin, a theory independent derivation of the fractional charge. An adiabatic local charge will not be the same because of the different screening properties of the FQHE states.

Fractional charge by several methods. Even though the excitations of all FQHE states are composite fermions, their excitations of charge $e^*$ of the quasihole following an argument put forth by W. P. Su. Let us specialize to $\nu = n/(2m + 1)$ and assume that $e_\phi$ is a collection of an integral number of elementary excitations of charge $e^*$. It is also reasonable to demand that an integer number of elementary excitations make an electron. The largest value of $e^*$ consistent with these requirements is

$$|e^*| = \frac{e}{2m \pm 1}$$

Thus, the presence of a gap at a fractional filling factor not only implies fractional charge, but is a sufficiently powerful restriction to essentially fix the value of the charge. While this argument tells us the value of the local charge, it is too crude for a microscopic description of this object.

(The above argument is incomplete. We also need to show that a collection of $2m \pm 1$ elementary excitations has the same braid properties as an electron. That turns out to be the case at $\nu = n/(2m \pm 1)$, as shown in an exercise. It is not always true, however; for example, the braid consistency would tell us that at $\nu = 1/2$ the largest allowed value of local charge is $|e^*| = 1/4$.)

**Method 2:** The local charge of a CF-quasiparticle at $\nu = n/(2m + 1)$ can be ascertained by noting that it is the bound state of an electron and $2p$ vortices:

$$e^* = -e + 2pe_\phi = -e + \frac{2m}{2m + 1}e = -e \frac{2m + 1}{2m + 1}.$$  \hspace{1cm} (107)

For $\nu = n/(2m - 1)$, the local charge is obtained by the replacement $2p \rightarrow -2p$.

**Method 3:** Let us ask what happens to an incompressible FQHE system when an electron is added to it. Let us consider the ground state at $\nu = n/(2m + 1)$ in the spherical geometry, which maps into $n$ filled $\Lambda$ levels of composite fermions. I.e., the effective flux

$$Q^* = Q - p(N - 1)$$

is such that the total number of single particle states in the lowest $n$ $\Lambda$ levels is precisely $N$. Now add one electron to this state **without changing the real external flux $Q$**. The new state of $N' = N + 1$ particles corresponds to a modified effective flux

$$Q'^* = Q - p(N' - 1) = Q^* - p.$$  \hspace{1cm} (109)

At $Q'^*$ the degeneracy of each $\Lambda$ level is reduced by $2p$ compared to $Q^*$. Thus $2p$ composite fermions from each of the $n \Lambda$ levels must be pushed up to the $(n + 1)^{st}$ $\Lambda$ level. Including the added particle, the $(n + 1)^{st}$ $\Lambda$ level has $2m + 1$ CF-quasiparticles. These are identical, can move independently, and can be removed far from one another. Since a net charge $-e$ was added, each CF-quasiparticle must carry an excess charge of precisely

$$-e^* = -e \frac{2m + 1}{2m + 1}.$$  \hspace{1cm} (110)

The additional electron thus creates a correlation hole around it while turning into a composite fermion, which forces $2m + 1$ fractionally charged objects. (Of course, the full state is still composed of unit charge electrons.)
**Method 4:** A direct method is to take the wave function of a single excitation, and obtain its local charge by direct integration of its density. While this does give the correct value, it does not bring out why the charge would not change upon variations in the wave function. The previous counting arguments are more powerful in that respect.

How do we know that the objects with charge $e^* = e/(2pn \pm 1)$ are indeed elementary? If they were not, the whole structure of the above theory would fall apart; in particular, the counting of the low energy states would change, which would not be consistent with either exact results or experiments.

### F. Fractional braid statistics

Halperin first realized that the quasiparticles and quasiholes of the FQHE obey fractional braid statistics, which was confirmed in a microscopic calculation by Arovas, Schrieffer and Wilczek for the quasihole at $\nu = 1/3$. The Berry phase associated with a closed loop of a CF-quasiparticle encircling an area $A$ is given by Eq. (78). The *average* change in the phase due to the insertion of another CF quasiparticle inside the loop is given by

$$
\Delta \Phi^* = 2\pi 2p \Delta \langle N_{\text{enc}} \rangle = 2\pi \cdot 2p \cdot \frac{e^*}{e} = 2\pi \frac{2p}{2pn + 1}.
$$

(111)

We have used that the change in the average number of electrons due to an extra CF-quasiparticle is $\Delta \langle N_{\text{enc}} \rangle = e^*/e = 1/(2pn + 1)$, which is valid so long as the two CF-quasiparticles do not overlap at any point of the closed trajectory. This path independent phase is interpreted in terms of fractional braid statistics. With $\Delta \Phi^* = 2\pi \alpha$, the braid statistics parameter is the product of the local charge and the number of vortices bound to a CF quasiparticle:

$$
\alpha_{\text{qp}} = 2p \cdot \frac{e^*}{e} = \frac{2p}{2pn + 1}.
$$

(112)

Essentially, even though each electron carries an even number of vortices, the expectation value of the electron number inside the loop has changed by a fraction, which amounts to the addition of a fractional number of vortices, thus producing a fractional braid statistics. The fractional braid statistics is a direct descendant of the fractional local charge.

### G. 5/2 and other fractions in the second LL

The matrix elements of the Coulomb interaction are different in different LLs. Because of the insensitivity of the FQHE on the interaction, one might expect that the FQHE in the second LL should have the same origin as that in the LLL, with the wave functions of the two related by a simple change of the LL index. However, the following facts indicate that the physics in the second LL requires further conceptual input:

- A FQHE state is observed at the half filled second LL ($\nu = 5/2$) as opposed to a compressible CF Fermi sea at the half filled LLL.
- Many fewer fractions are observed in the second LL, which are also more delicate than those in the LLL, requiring higher quality samples and lower temperatures.
- The other observed fractions, such as $2 + 1/3, 2 + 2/5,$ etc. correspond to those seen in the LLL. However, computer calculations show that the ground states at these fractions have poor overlaps with $\Psi_{\text{CF}}$, indicating physics beyond weakly interacting composite fermions.

It is appealing to attempt to model the second LL FQHE states in terms of *interacting* composite fermions. For $\nu = 5/2$, Moore and Read have proposed the following wave function for the ground state:

$$
\Psi_{\text{Pf}}^{5/2} = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i < j} (z_i - z_j)^2 \exp \left[ -\frac{1}{4} \sum_k |z_k|^2 \right]
$$

(113)

The Pfaffian of an antisymmetric matrix $M$, defined as

$$
\text{Pf}(M_{ij}) = A(M_{12} M_{34} \cdots M_{N-1,N}),
$$

(114)
where $A$ is the antisymmetrization operator, is nothing but the BCS wave function for sully spin polarized electrons:

$$\Psi_{\text{BCS}} = A \left[ \phi_0(r_1 - r_2) \phi_0(r_3 - r_4) \cdots \phi_0(r_{N-1} - r_N) \right]$$

(115)

with $M_{ij} = 1/(z_i - z_j)$ identified with the pair wave function $\phi_0(r_i - r_j)$. Because the Jastrow factor composite-fermionizes the Pfaffian in Eq. 113, $\Psi_{1/2}^{\text{Pf}}$ is interpreted as the p-wave paired state of composite fermions carrying two vortices. The quasiparticles of this state are believed to obey non-Abelian braid statistics, which is a subject of current theoretical and experimental investigation. Various interesting proposals have been advanced for the other second LL FQHE states.

H. CF Wigner crystals

Prior to the discovery of the FQHE, the expectation was that once the kinetic energy degree of freedom is quenched by forcing all electrons into the LLL, the interaction energy would govern the physics and induce a WC. Experiments have showed, however, that in a range of filling factors, the interaction energy favors the formation of composite fermions instead, which form a liquid state. Explicit calculations indicate that even for very small $\nu$, the CF WC has lower energy than the electron WC; here electrons capture fewer than the maximum number of vortices available to them, using the remaining degrees of freedom to form a crystal. It is not known what property would decisively distinguish the CF WC from the electron WC. The composite fermions in the partially filled AL in the vicinity of $\nu^* = n$ can also form a WC.

Exercises

♣ The form in Eq. 80 suggests a generalized class of wave functions (Jain 1989) $\Psi_{\nu} = P_{\text{LLL}} \prod_{j=1}^{m} \Phi_{n_j}$, where $n_j$ are integers, $\Phi_n$ denotes the Slater determinant wave function of $n$ filled Landau levels, and $m$ is an odd integer to ensure antisymmetry for $\Psi$. (These wave functions in general do not have a CF interpretation.) Obtain the relation between the filling factor $\nu$ and the integers $n_j$. What excitation has the smallest local charge? Construct a state at $\nu = 1/2$ and determine the magnitude of the smallest local charge. (If this state could be realized, its excitations would obey nonabelian braid statistics.)

♣ Obtain the relative braid statistics of a quasiparticle going around a quasihole. Assuming that an exciton produces no statistical phase, deduce the braid statistics for quasiholes. Finally, show that a bound state of $2m \pm 1$ quasiparticles has the same charge and braid statistics as an electron.

♣ Here is yet another derivation of fractional local charge. We will use the result that destroying an electron at $\eta$ in the state $\Psi(r_1, \cdots, r_{N-1}, r_N)$ produces, neglecting normalization factors, $\Psi(r_1, \cdots, r_{N-1}, \eta)$; i.e., it replaces one of the electron coordinates by $\eta$ to create a charge deficiency of one (in units of the electron charge) at $\eta$. For example, destroying an electron at $\eta$ in $\Psi_{1/m}^{\text{Laughlin}}$ (Eq. 55) produces $\prod_{j=1}^{N-1} (z_j - \eta)^m \Psi_{1/m}^{\text{Laughlin}}(z_1, \cdots, z_{N-1})$, which shows that a single quasihole $\prod_{j=1}^{N-1} (z_j - \eta)^m \Psi_{1/m}^{\text{Laughlin}}$ has a charge deficiency of $1/m$. Show that: (i) the replacement $r_N \rightarrow \eta$ in $\Phi_n$ produces a single hole, and (ii) multiplying $\Phi_n$ by $\prod_{j}(z_j - \eta)$ produces $n$ holes. (iii) Finally, determine how many CF-quasiholes are created by the replacement $r_N \rightarrow \eta$ in $\Psi_{1/m}^{\text{CF}}$ and deduce the charge of a single CF-quasihole.
VIII. MISCELLANEOUS REMARKS

• Previous quantum liquids, such as the $^4$He superfluid, the $^3$He superfluid, or a BCS superconductor, are all Bose Einstein condensates. The FQHE is not a BEC. The FQHE state has no spontaneously broken symmetry, no off-diagonal-long range order, and no order parameter in the sense of Landau. It constitutes a distinct paradigm for macroscopic quantum behavior. It is a quantum fluid, because quantum mechanical phases play a central role in determining its macroscopic behavior, but these phases enter in the form of quantized vortices, which are captured by electrons to produce new particles that experience a reduced effective magnetic field.

• The FQHE state is a topological state. This is clarified from the observation that composite fermions are themselves topological particles because the vortex is a topological object: a closed loop around a vortex produces a phase of $2\pi$ independent of the shape or the size of the loop. (This is why we can count the number of vortices.) All states of composite fermions are thus topological. In fact, the FQHE is doubly topological. The number of filled ALs can also be thought of as a topological integer, by analogy to the number of filled LLs that can be interpreted as a topological invariant, namely the first Chern class of a U(1) principal fiber bundle on a torus.

• The FQHE state is an example of the general principle (“duality”) that a strongly interacting system of one kind of particles is equivalent to a weakly interacting system of another kind of particles. In other words, nature eliminates degeneracy by producing new emergent particles.

• While composite fermions and their ALs are analogous to electrons and LLs, there is a conceptual difference. The LLs can be derived at the level of a single electron, but such a derivation is not possible for ALs. The concept of a single composite fermion itself is nonsensical, as there can be no vortex without other electrons. The existence of composite fermions and their ALs can be established only by comparing their consequences to the solutions of the intrinsically many body problem.

• It is natural to interpret the energy gap of the state at $\nu^* = n$ as an effective cyclotron energy $\hbar eB^*/m^*c$ for composite fermions. The CF mass defined in this manner is roughly filling factor independent along a given sequence. This mass $m^*$ has no relation to the electron band mass; it arises purely due to interactions.

• A mass is generated due to interactions in a theory without mass: the “massless” electrons turn into massive composite fermions by swallowing flux quanta.
IX. FREQUENTLY ASKED QUESTIONS

This section addresses some questions that I have often encountered on this topic.

i. In what sense are composite fermions particles?

A vortex $\prod_{j=1}^{N} (z_j - \eta)$ is an emergent collective variable in which all electrons participate. Each composite fermion is thus made up of all electrons. In spite of this complexity, experiments have shown that composite fermions are legitimate particles in the same sense as other emergent particles of condensed matter (e.g. phonons, magnons, Cooper pairs, or Landau quasiparticles): they are weakly interacting objects with well defined charge, spin, statistics and other nice properties that we associate with particles, and they exhibit many phenomena that are expected of any self-respecting fermionic particles, such as a Fermi sea, cyclotron orbits, Shubnikov de Haas oscillations, IQHE, FQHE, pairing, excitons, etc.

ii. Why must we think about higher Landau levels in a theory of electrons in high magnetic fields?

While the LLL restriction is convenient for theory, it is not necessary for the phenomenon of the FQHE. LL mixing is, in fact, always present in experiments, and causes no measurable correction to the Hall quantization, which is an experimental proof that the phase diagram of the FQHE state contains regions with some amount of LL mixing. The job of theory is to identify a point inside the FQHE phase where the physics is the simplest, and approach the physical point perturbatively starting from there. Because we do not know a priori where the physics would be most transparent, it is best not to disallow LL mixing at the very outset.

Within the LLL space, at filling $\nu = 1/m$ we have a simple prescription due to Laughlin: here the number of vortices per electron, $\nu^{-1}$, is precisely an odd integer, and attaching all of them to electrons, as allowed by antisymmetry, gives a unique solution. For other values of $\nu$, after we have tied an odd number of vortices to each electron (only the Pauli vortex is allowed for $\nu > 1/3$), we are clueless about what to do with the remaining vortices. What the CF theory has shown is that “unique” wave functions can be constructed for a larger set of fractions $\nu = n/(2pn \pm 1)$ by allowing some amount of LL mixing; these wave functions become very complicated when projected into the LLL, and no simple structure emerges for vortices, but they demonstrably provide an excellent representation of the actual eigenstates. In other words, while the FQHE wave functions in the LLL are very complicated, they are adiabatically connected to some simple wave functions that involve some admixture with higher LLs, which, in turn, allow us to “read off” the physics of composite fermions in an effective magnetic field.

Experiments have shown that the analogy to higher LLs is not just a mathematical trick but contains the essential physics of the FQHE. It is a fact that the low-energy spectrum of interacting electrons at $\nu$, as determined in computer experiments, resembles that of weakly interacting fermions at $\nu^*$ (which is in general greater than 1), and that the FQHE resembles IQHE when plotted as a function of the effective magnetic field. That makes the analogy to higher LLs rather natural and also physical.

We finally note that the idea of extending the Hilbert space in search of a simple solution is a standard one. In many instances, such as large $N$, large $S$, slave bosons and Schwinger boson formulations, one extends the Hilbert space into even unphysical directions in search for guidance, but of course, the theory must always be projected back into the physical sector at the end of the day.

iii. How do we know that the repulsive correlations built in the unprojected wave functions of Eq. 73 survive projection?

Explicit calculations have shown that even without LLL projection, the wave functions of the type given in Eq. 73 are predominantly in the LLL. A simple way of seeing why that might be true is to notice that while the factor multiplying the gaussian is not analytic (as required for a LLL wave function), it is “almost analytic,” in that it contains up to $O(N)$ powers of $z$ whereas only up to $n$ powers up to $\bar{z}$, where $n$ is the number of occupied $\Lambda$ levels. As a result, one may expect that LLL projection might not change the correlations drastically. Nonetheless, the real proof comes only from comparing the LLL projected wave functions against the Coulomb eigenstates, which shows that they do as good a job of keeping repulsive correlations as possible within the phase space constraints of the LLL.

iv. Why don’t electrons bind an even number of vortices?

The bound state of an electron and an odd number of vortices is a boson. Which bound state is formed is often a complicated question, the resolution of which requires a detailed consideration of energetics and comparison with experiments, which clearly show that electrons bind an even number of vortices. An insight into this fact can be gained from the observation that when our goal is to build correlations that keep particles apart, fermions
are favored over bosons because fermions feel the Pauli repulsion even in the non-interacting limit; there is no IQHE for non-interacting bosons. In fact, even when interacting bosons are subjected to a strong magnetic field, they capture an odd number of vortices to turn into composite fermions. (In other words, their state is described by the wave function in Eq. 74 with $2p$ replaced by an odd integer.)

v. **What is the size of a composite fermion?**

The density profile of a CF quasiparticle can be evaluated from the CF theory (see Fig. 8 for example), from which its size can be estimated. The size of the CF quasiparticle at $\nu = n/(2pn \pm 1)$ is roughly equal to the size of a localized electron in the $(n+1)^{th}$ LL, and is thus governed by the effective magnetic length $\ell^* = \sqrt{\hbar c/e|B^*|}$ rather than the real magnetic length, diverging as we approach $1/2$.

vi. **Do composite fermions provide an effective theory of the FQHE?**

Because condensed matter systems are very complicated, it is often not possible to write a fully quantum mechanical solution of the original Hamiltonian, and one resorts to effective descriptions. The wave functions $\Psi_{\text{CF}}$, however, are not written in terms of some effective coordinates but in terms of electron coordinates; they represent the actual electron wave functions of the FQHE in the quantum chemistry sense. It is a remarkable feature of the FQHE that such a truly microscopic description is available for a strongly correlated state with remarkable properties.

vii. **Isn’t it a serious problem that the CF wave functions do not depend on the interaction parameters?**

One might have thought so at first, because a variational parameter would have given us a knob which we could tune to improve the solution. However, given how close $\Psi_{\text{CF}}$ are to the exact solution, their parameter free nature actually turns into a virtue, because it allows a test of the theory without any “fudging.” The reader might be relieved to know that when we improve $\Psi_{\text{CF}}$ by allowing for $\Lambda L$ mixing, the improved wave function will depend, albeit only slightly, on the form of the electron-electron interaction.

viii. **Why are the incompressible FQHE wave functions so insensitive to the actual form of the interaction?**

The fact that the FQHE wave functions are insensitive to the form of the interaction is seen in exact diagonalization studies, and is indeed a surprising, though fortunate, aspect of the FQHE. Some insight into it can be gained from the CF theory by analogy to the more obvious fact that the IQHE wave functions are weakly dependent on interaction provided it is weak compared to the cyclotron energy gap. The robustness of the FQHE wave functions is an evidence that the residual interaction between composite fermion is weak compared to the energy gap.

ix. **What is the Hamiltonian for composite fermions?**

The Coulomb Hamiltonian in the lowest Landau level. We have seen that $\Psi_{\text{CF}}$ are a faithful representation of the low-energy Coulomb eigenstates. Numerical calculations show that any Hamiltonian with sufficiently strong short range interaction produces composite fermions in the LLL.

x. **Is there a Hamiltonian for which $\Psi_{\text{CF}}$ are exact?**

No simple Hamiltonian with such property is known. Such a Hamiltonian exists for Laughlin’s wave function which is simple, but in general $\Psi_{\text{CF}}$ are so complicated that it is unlikely, in my opinion, that a simple Hamiltonian will be found for which they are the exact solutions.

It must be noted, however, that the confirmation of theory comes not from such Hamiltonians but its comparison to the solutions of the Coulomb interaction. Wave functions are known which are exact solutions for certain Hamiltonians, but do not apply to the real world. Also, such model Hamiltonians for FQHE are not solvable in any real sense. For example, the model Hamiltonian for which Laughlin’s wave function is exact was constructed after the wave function was known, and the only thing we know about this Hamiltonian is that Laughlin’s wave function is its exact zero energy ground state; no non-zero energy eigenstates of this Hamiltonian can be derived.

xi. **How can the CF quasiparticles be charge-one composite fermions and fractionally charged anyons at the same time?**

The CF quasiparticles are of course what they are – we know their precise microscopic wave functions without any ambiguity. However, apparently different physical interpretations arise when viewed from different perspectives, that is, relative to different reference states, also called “vacua.” Consider the state at $\nu^* = n + \epsilon$. If we take our vacuum to be the state with no particles, then all composite fermions are charge one fermions, with a wave function that is antisymmetric under the exchange of any two composite fermions. Indeed, the physics of the IQHE of composite fermions or their Fermi sea relies fundamentally on their fermionic statistics. On the
other hand, if we take the incompressible state at $\nu^* = n$ as the reference state, we must “integrate out” the composite fermions in the filled $\Lambda$Ls to obtain an effective description in terms of only the composite fermions in the $(n + 1)^{\text{th}} \Lambda$ level. This produces fractionally charged objects that have fractional braid statistics. The reason is that, unlike in the IQHE, the vacuum is not a passive spectator; a composite fermion in the $(n + 1)^{\text{th}} \Lambda$ level sees $2p$ vortices on all other composite fermions, including those we have decided to incorporate into the vacuum. (The Jastrow factor in Eq. 74 contains all CF coordinates.) The fractional charge arises from a screening of the unit charge by the vacuum, and the fractional braid statistics encapsulates the average effect of the Berry phases arising from the composite fermions in the vacuum.

xii. Is there a small parameter for composite fermions? Can we perturbatively improve the CF theory?

While there is no small parameter for electrons in the FQHE regime, a small parameter can be identified for composite fermions. Either the interaction between composite fermions or the $\Lambda$L mixing can be treated as a small parameter; these two are related because weak interaction between composite fermions implies small $\Lambda$L mixing. Experiments tell us that the interaction between composite fermions is weak, because a large fraction of the phenomenology can be explained by assuming that composite fermions are not interacting at all. The fact that the wave functions $\Psi_{\nu}^{\text{CF}}$, which neglect $\Lambda$L mixing, are almost exact (for a range of parameters in the lowest LL) also demonstrates that $\Lambda$L mixing is negligible.

If needed, $\Psi^{\text{CF}}$ can be systematically improved by successively allowing mixing with higher and higher $\Lambda$Ls and performing CF diagonalization in a larger CF basis.

While the interaction between composite fermions is always much smaller than that between electrons, it can sometimes have non-perturbative consequences, such as fractional quantum Hall effect or pairing of composite fermions.

xiii. How can pairing arise in a model without attractive forces?

There is a strong Coulomb repulsion between electrons, but we must remember that the objects that pair up are not electrons but composite fermions. The interaction between composite fermions is weak, and there is no reason, in principle, why it can sometimes not be weakly attractive. The Coulomb interaction is screened through the binding of vortices to electrons, which creates a correlation hole around each electron. Since the number of bound vortices cannot change continuously, it should not be surprising that such a binding can overscreen the Coulomb interaction for certain parameters, which seems to be what is happening at $5/2$. We note that Kohn and Luttinger have shown that even ordinary electrons in a structureless jellium background (with no phonons) subject to the Coulomb repulsion are ultimately unstable to pairing due to screening of the Coulomb interaction that produces attractive interaction in some large angular momentum channel.

xiv. What is the origin of the odd-denominator rule?

There is no fundamental principle of nature that precludes FQHE at even denominator fractions, as proven by the observation of $5/2$. Most observed fractions have odd denominators because non-interacting composite fermions produce only odd-denominator fractions. Even denominator fractions can occur only because of the weak residual interaction between composite fermions, and are therefore expected to be much weaker.

xv. Won’t all fractions eventually be observed with purer samples and lower temperatures?

No. Even though one can construct candidate incompressible states for all odd and even denominator fractions, the Coulomb interaction will realize but a finite number of them. Many examples where FQHE could occur but does not are already known: the FQHE state is not relevant in high LLs (a Hartree-Fock stripe or bubble crystal is energetically preferred) or at very low fillings in the LLL (where the Wigner crystal is favored); it is also quite likely that no states other than $n/(2n \pm 1)$ occur for $\nu^* \geq 3$, given that only integers are observed even for electrons beyond the third LL.
X. FURTHER READING

4 “Composite Fermions in the Fractional Quantum Hall Effect,” H.L. Stormer and D.C. Tsui in Ref. 3.
8 “Composite fermions – experimental findings,” R. L. Willett in Ref. 2.